

An Integrable $U_q(\widehat{gl}(2|2))_1$ -Model: Corner Transfer Matrices and Young Skew Diagrams

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The path space of an inhomogeneous vertex model constructed from the vector representation of $U_q(gl(2|2))$ and its dual is studied for various choices of composite vertices and assignments of $gl(2|2)$ -weights. At $q = 0$, the corner transfer matrix Hamiltonian acts trigonally on the space of half-infinite configurations subject to a particular boundary condition. A weight-preserving one-to-one correspondence between the half-infinite configurations and the weight states of a level-one module of $U_q(\widehat{sl}(2|2))/\mathcal{H}$ with grade $-n$ is found for $n \geq -3$ if the grade $-n$ is identified with the diagonal element of the CTM Hamiltonian. In each case, the module can be decomposed into two irreducible level-one modules, one of them including infinitely many weight states at fixed grade. Based on a mapping of the path space onto pairs of border stripes, the character of the reducible module is decomposed in terms of skew Schur functions. Relying on an explicit verification for simple border stripes, a correspondence between the paths and level-zero modules of $U_q(\widehat{sl}(2|2))$ constructed from an infinite-dimensional $U_q(gl(2|2))$ -module is conjectured.

Keywords: Integrable models, quantum affine superalgebras, corner transfer matrices, Young skew diagrams

I. INTRODUCTION

A link between integrable lattice models and irreducible modules of quantum affine algebras $U_q(\widehat{g})$ is provided by the spectra of the corner transfer matrix originally introduced in [1]. The trace of the CTM of a homogeneous integrable vertex model based on $U_q(\widehat{g})$ relates to an affine Lie algebra character [2,3]. Incorporating the concept of vertex operators, a mathematical characterization of physical objects including transfer matrices and N-point correlators has been developed [4,5]. Descriptions of the same type have been given for $U_q(\widehat{sl}(2))$ -vertex models built from two different spin representations [6,7] and for other integrable models (see [8–11] and references in [5]). More recently, the CTM spectrum of an inhomogeneous vertex model based on the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$ has been investigated [12]. An analysis of all CTM-eigenvectors with eigenvalue ≥ -4 suggests a one-to-one correspondence between the space of half-infinite configurations subject to a suitable boundary condition and the level-one module $V(\Lambda_2)$ of $U_q(\widehat{sl}(2|1))$.

This study deals with an integrable vertex model characterized by alternating sequences of the vector representation W of $U_q(gl(2|2))$ and its dual W^* in horizontal as well as vertical direction. The R-matrices attributed to composite vertices are required to provide well-defined invertible maps of the tensor products of evaluation modules in the limit $q \rightarrow 0$. For two choices of composite vertices, this is achieved by a suitable adjustment of the inhomogeneity in the spectral parameters. In the limit $q \rightarrow 0$, the action of the corner transfer matrix Hamiltonian in the space of the half-infinite configurations becomes triangular with respect to the canonical basis. Moreover, the diagonal elements decouple into a contribution depending only on the part governed by the modules W and into a second contribution due to the modules W^* only. Thus the present model clearly differs from the $U_q(\widehat{sl}(2|1))$ -model in [12] whose CTM Hamiltonian does not show a triangular action in the canonical basis even for $q \rightarrow 0$. For either choice of the composite vertices, two assignments of $U_q(gl(2|2))$ -weights are considered. With respect to two assignments, the $U_q(gl(2|2))$ -weights of all configurations with the diagonal element of the CTM Hamiltonian given by $-n$ are collected for $n = 0, 1, 2, 3$. They are found in one-to-one correspondence with the weight states of a reducible level-one module $\tilde{V}(\Lambda_0)$ at grade $-n$. Two reducible modules $\tilde{V}(\Lambda_1 + \Lambda_4)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ account for the remaining assignments of weights. All are modules of $U_q(\widehat{sl}(2|2))/\mathcal{H}$ with \mathcal{H} denoting the center of the algebra. Relying on these results, the correspondence may be assumed to hold at any grade. Each of the reducible modules can be decomposed into two irreducible level-one modules, one of them containing a finite-dimensional, the other an infinite-dimensional $U_q(\widehat{sl}(2|2))$ -module as their grade-zero subspace. Hence the reducible modules are nonintegrable. The occurrence of nonintegrable modules appears to be a typical feature of inhomogeneous vertex models associated to $U_q(\widehat{gl}(m|m'))$

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with $m, m' > 1$. Relations between $\tilde{V}(\Lambda_0)$, $\tilde{V}(\Lambda_1 + \Lambda_4)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ and their decompositions are readily inferred from boson realizations of the reducible modules. All subsequent investigations specialize to $\tilde{V}(\Lambda_0)$.

Skew Young diagrams have been employed in various related contexts. They label particular modules of the Yangian $Y(sl(N))$ called the tame modules [13,14]. The $\widehat{sl}(N)$ -character of an irreducible tame module can be expressed as a product of skew Schur functions [14–16]. The latter are involved in the spectral decomposition of the path space of $U_q(\widehat{sl}(N))$ -vertex models with respect to an infinite family of mutually commuting Hamiltonians defined by the local energy function [16–18]. Taking advantage of a one-to-one correspondence between the paths and a suitable class of skew Young diagrams, the characters specifying the degeneracy of the spectrum of level-one models can be written in terms of skew Schur functions. Identification of these characters with the characters of irreducible tame modules yields a link to the $Y(sl(N))$ -module structure of the level-one integrable modules of $\widehat{sl}(N)$ underlying the path space of the model. Skew Young diagrams also label the decomposition of integrable modules at higher levels [18,19]. For integrable level-one $\widehat{sl}(2)$ -modules, the action of the Yangian algebra and the resulting $Y(sl(2))$ -structure have first been described in [20] and [21] (further references on Yangian module structures are found in [19]). Integrable level-one modules of the quantum affine algebras $U_q(\widehat{sl}(N))$ and $U_q(\widehat{gl}(N))$ show an analogous structure emerging under a level-zero action defined in terms of Cherednik operators [22–29]. The analysis in [26] exploits correspondences between skew Young diagrams of border strip type and irreducible level zero modules. These modules are given as subspaces of tensor products of evaluation modules obtained from the vector representation of $U_q(sl(N))$.

The paths of the $U_q(\widehat{gl}(2|2))$ -model studied here correspond to pairs of border stripes, one of them related to the component of the path contributed by the W -modules and the other to the W^* -component. Each semi-standard super tableau of a finite border strip subject to a particular condition on the labeling of the lowest leftmost box is mapped onto the W -part of a configuration and vice versa. The full set of semi-standard super tableaux for a given border strip with M boxes corresponds to a specific subspace of $W_{x_1} \otimes W_{x_2} \otimes \dots \otimes W_{x_M}$. Here W_{x_i} denote evaluation modules associated to W with the ratios of the spectral parameters x_i determined by the shape of the border strip.

Similarly, W^* -path components may be attributed to semi-standard super tableaux of a finite border strip. Two conditions are imposed on the length and height of the two border stripes combined for the labeling of paths. Such pairs of semi-standard super tableaux map onto a subspace of the paths provided that the tableaux related to the W -component fulfill the condition mentioned above. This subspace in turn contains a subspace of paths corresponding to the weight states of the irreducible level-one module $V(\Lambda_0)$. To each of the pairs of border stripes, a tensor product $W_{x_1} \otimes W_{x_2} \otimes \dots \otimes W_{x_M} \otimes W_{\bar{x}_1}^* \otimes W_{\bar{x}_2}^* \otimes \dots \otimes W_{\bar{x}_M}^*$ is assigned. Here the ratios of the spectral parameters \bar{x}_i of the evaluation modules $W_{\bar{x}_i}^*$ follow from the shape of the second border strip. The ratio x_1/\bar{x}_1 is fixed by the requirement that $\Delta^{(2M-1)}(H_s^1 + H_s^3)$ annihilates the tensor product for any $s \neq 0$. An examination of pairs of border stripes with simple shapes suggests that the weight states of $V(\Lambda_0)$ are in one-to-one correspondence with certain subspaces of the tensor products.

To extend the map between semi-standard super tableaux and half-infinite configurations to the complete path space of the model, a particular type of infinite border stripes is introduced. The set of all semi-standard super tableaux of these border stripes exactly corresponds to the entire space of W^* -path components. Based on this correspondence, a character decomposition of $\tilde{V}(\Lambda_0)$ similar to the one presented in [16] for the $U_q(\widehat{sl}(N))$ -models is given. This expression can be expected to reflect a decomposition of the level-one module $\tilde{V}(\Lambda_0)$ into modules of a level-zero action. As a preparation of such an analysis, level-zero modules of $U_q(\widehat{sl}(2|2))$ matching the entire configuration space are constructed. Particular $U_q(\widehat{sl}(2|2))$ -modules are introduced as tensor products of evaluation modules V_{y_i} obtained from an infinite-dimensional $U_q(\widehat{gl}(2|2))$ -module V . This class of tensor products is supplemented by a one-dimensional level-zero module of $U_q(\widehat{sl}(2|2))$. The $U_q(\widehat{gl}(2|2))$ -weights present in these $U_q(\widehat{sl}(2|2))$ -modules correspond to the W^* -path components combined with the unique W -path component with vanishing diagonal element of the corner transfer matrix Hamiltonian. Suitable tensor products for general diagonal elements involve evaluation modules $W_{\bar{x}_i}^*$ in addition. Then tensor products of evaluation modules W_{x_i} , $W_{\bar{x}_i}^*$ and V_{y_i} with a specific prescription for the ratios of the spectral parameters are associated to pairs of border stripes. Certain classes of $U_q(\widehat{sl}(2|2))$ -modules are introduced as parts of these tensor products closed under the evaluation action or as subquotients. Based on an explicit verification for border stripes with a simple form, it is conjectured that the $U_q(\widehat{gl}(2|2))$ -weights found in the level-zero modules related to a pair of border stripes exactly correspond to the weights of the configurations attributed to the same pair.

The new findings presented here can be summarized as follows:

1. Reducible nonintegrable $U_q(\widehat{gl}(2|2))$ -modules with level one are related to the corner transfer matrices of an inhomogeneous vertex model based on this algebra. The CTM acts on the space Ω_A of half-infinite configurations subject to a particular boundary condition. Each of the level-one modules can be decomposed into two irreducible

level-one modules, one of them weakly integrable. Two generalized energy functions attributed to the R -matrices give rise to a quasiparticle character formula for the reducible modules.

2. The entire space Ω_A of half-infinite configurations can be mapped onto a particular subset of all semi-standard super tableaux of a pair of border stripes and vice versa. Each pair consists of a finite and an infinite border strip. The map allows to express the character of the reducible level-one module in terms of skew Schur functions. Both character formulae for the reducible module split into a factor exclusively due to the W -components and another factor arising from the W^* -components. One or two level-zero module(s) are assigned to each pair of border stripes. Their $U_q(\widehat{gl}(2|2))$ -weights coincide with the weights of all configurations related to the same pair of border stripes. Partial results are obtained for the irreducible weakly integrable submodule.

All results are obtained by examination of weight structures at grades with low absolute values. Assuming that the results are valid at all grades, the above statements are formulated in terms of four conjectures found in Sections IV and at the end of Section VII, VIII A, VIII B.

Section II collects definitions of the quantum affine superalgebra and the vector representation. Then the construction of the vertex model is described. Section III proceeds with an analysis of the spectra of the CTM Hamiltonian. Section IV deals with the mapping onto the level-one modules and Section V presents boson realizations for all level-one modules involved. In Section VI, the border stripes relevant to the $U_q(\widehat{gl}(2|2))$ -model and their semi-standard super tableaux are specified. $U_q(\widehat{sl}(2|2))$ -modules related to them are constructed by means of the evaluation modules W_{x_i} or, alternatively, of $W_{\bar{x}_i}^*$. Both types of modules are combined in Section VII, where their relation to the irreducible level-one module $V(\Lambda_0)$ is discussed for grades ≥ -3 . Section VIII provides a character formula for $\tilde{V}(\Lambda_0)$ in terms of skew Schur functions. Introducing the infinite-dimensional $U_q(\widehat{gl}(2|2))$ -module V , the $U_q(\widehat{sl}(2|2))$ -modules involving W_{x_i} , $W_{\bar{x}_i}^*$ and V_{y_i} are characterized.

II. THE MODEL

The basic construction of the vertex model is covered by the quantum affine superalgebra $U'_q(\widehat{sl}(2|2))$. Defining relations for affine Lie superalgebras [30] and quantum affine superalgebras with each choice of simple roots can be found in [31]. In terms of its Chevalley basis, $U'_q(\widehat{sl}(2|2))$ is introduced as the associative \mathbb{Z}_2 -graded algebra over $\mathbb{C}[[q-1]]$ generated by $e_i, f_i, q^{\pm h_i}$, $i = 0, 1, 2, 3$, through the defining relations

$$\begin{aligned} q^{h_i} q^{h_j} &= q^{h_j} q^{h_i} & q^{h_i} e_j q^{-h_i} &= q^{a_{ij}} e_j & q^{h_i} f_j q^{-h_i} &= q^{-a_{ij}} f_j \\ [e_i, f_j] &= \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \\ [e_1, e_3] &= [e_0, e_2] = [f_1, f_3] = [f_0, f_2] = 0 \end{aligned} \tag{1}$$

$$\begin{aligned} [e_0, e_1]_q, [e_0, e_3]_q &= 0 & [f_0, f_1]_{q^{-1}}, [f_0, f_3]_{q^{-1}} &= 0 \\ [e_1, e_2]_q, [e_1, e_0]_q &= 0 & [f_1, f_2]_{q^{-1}}, [f_1, f_0]_{q^{-1}} &= 0 \\ [e_2, e_1]_q, [e_2, e_3]_q &= 0 & [f_2, f_1]_{q^{-1}}, [f_2, f_3]_{q^{-1}} &= 0 \\ [e_3, e_2]_q, [e_3, e_0]_q &= 0 & [f_3, f_2]_{q^{-1}}, [f_3, f_0]_{q^{-1}} &= 0 \end{aligned} \tag{2}$$

The matrix elements a_{ij} in (1) depend on the choice of simple roots. With all simple roots odd, the matrix a is given by

$$a = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \tag{3}$$

and the \mathbb{Z}_2 -grading assigns the value 1 to all e_i, f_i and the value 0 to $q^{\pm h_i}$. In equation (1), $[\cdot, \cdot]$ denotes the Lie superbracket $[x, y] = xy - (-1)^{|x| \cdot |y|} yx$. The q -deformed superbrackets in the Serre relations (2) are defined by

$$[e_i, e_j]_q = e_i e_j + q^{a_{ij}} e_j e_i$$

$$[f_i, f_j]_{q^{-1}} = f_i f_j + q^{-a_{ij}} f_j f_i \quad (4)$$

$U'_q(\widehat{sl}(2|2))$ can be equipped with a graded Hopf algebra structure introducing the coproduct

$$\Delta(e_i) = q^{h_i} \otimes e_i + e_i \otimes 1 \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i \quad \Delta(q^{\pm h_i}) = q^{\pm h_i} \otimes q^{\pm h_i} \quad (5)$$

the antipode

$$S(e_i) = -q^{-h_i} e_i \quad S(f_i) = -f_i q^{h_i} \quad S(q^{\pm h_i}) = q^{\mp h_i} \quad (6)$$

with the property $S(xy) = (-1)^{|x||y|} S(y)S(x)$ and the counit

$$\epsilon(e_i) = \epsilon(f_i) = \epsilon(h_i) = 0 \quad \epsilon(1) = 1 \quad (7)$$

The coproduct satisfies $\Delta(xy) = \Delta(x)\Delta(y)$ with the product operation defined by $(x_1 \otimes x_2)(y_1 \otimes y_2) = (-1)^{|x_2||y_1|} x_1 y_1 \otimes x_2 y_2$. Due to the special choice of simple roots, the antipode fulfills $S^2 = id$. Including the grading operator d with the properties

$$[d, e_i] = \delta_{i,0} e_i \quad [d, f_i] = -\delta_{i,0} f_i \quad [d, h_i] = [d, d] = 0 \quad (8)$$

yields the quantum affine superalgebra $U_q(\widehat{sl}(2|2))$. Coproduct and antipode of the grading operator are defined by $\Delta(d) = d \otimes 1 + 1 \otimes d$ and $S(d) = -d$.

$U'_q(\widehat{sl}(2|2))$ is a subalgebra of the quantum affine superalgebra $U'_q(\widehat{gl}(2|2))$ which may be introduced by means of L-operators or by Drinfeld generators. The latter are preferred here since the Drinfeld basis [41,31] underlies the free boson realizations of a quantum affine (super)algebra. For $U'_q(\widehat{sl}(m|m'))$, the Drinfeld basis has been provided in [31].

$U'_q(\widehat{gl}(2|2))$ is defined as the associative \mathbb{Z}_2 -graded algebra over $\mathbb{C}[[q-1]]$ with generators $E_n^{i,\pm}$, $i = 1, 2, 3$; $n \in \mathbb{Z}$ and $\Psi_{\pm n}^{j,\pm}$, $j = 1, 2, 3, 4$; $n \in \mathbb{Z}_+$ and the central element q^c . In terms of the generating functions

$$E^{i,\pm}(z) = \sum_{n \in \mathbb{Z}} E_n^{i,\pm} z^{-n-1} \quad \Psi^{j,\pm}(z) = \sum_{n \geq 0} \Psi_{\pm n}^{j,\pm} z^{\mp n} \quad (9)$$

the defining relations read

$$\begin{aligned} \Psi_0^{j,+} \Psi_0^{j,-} &= \Psi_0^{j,-} \Psi_0^{j,+} = 1 \\ \Psi^{j_1,\pm}(z) \Psi^{j_2,\pm}(w) &= \Psi^{j_2,\pm}(w) \Psi^{j_1,\pm}(z) \\ \Psi^{j_1,+}(z) \Psi^{j_2,-}(w) &= \frac{(z - wq^{c+a_{j_1,j_2}})(z - wq^{-c-a_{j_1,j_2}})}{(z - wq^{c-a_{j_1,j_2}})(z - wq^{-c+a_{j_1,j_2}})} \Psi^{j_2,-}(w) \Psi^{j_1,+}(z) \end{aligned} \quad (10)$$

$$\begin{aligned} \Psi^{j,+}(z) E^{l,+}(w) &= q^{a_{jl}} \frac{z - wq^{-\frac{c}{2}-a_{jl}}}{z - wq^{-\frac{c}{2}+a_{jl}}} E^{l,+}(w) \Psi^{j,+}(z) \\ \Psi^{j,+}(z) E^{l,-}(w) &= q^{-a_{jl}} \frac{z - wq^{\frac{c}{2}+a_{jl}}}{z - wq^{\frac{c}{2}-a_{jl}}} E^{l,-}(w) \Psi^{j,+}(z) \\ \Psi^{j,-}(z) E^{l,+}(w) &= q^{a_{jl}} \frac{z - wq^{\frac{c}{2}-a_{jl}}}{z - wq^{\frac{c}{2}+a_{jl}}} E^{l,+}(w) \Psi^{j,-}(z) \\ \Psi^{j,-}(z) E^{l,-}(w) &= q^{-a_{jl}} \frac{z - wq^{-\frac{c}{2}+a_{jl}}}{z - wq^{-\frac{c}{2}-a_{jl}}} E^{l,-}(w) \Psi^{j,-}(z) \end{aligned} \quad (11)$$

$$[E^{l_1,+}(z), E^{l_2,-}(w)] = \frac{1}{wz} \frac{\delta_{l_1,l_2}}{q - q^{-1}} \left(\sum_{n \in \mathbb{Z}} \left(q^c \frac{w}{z} \right)^n \Psi^{l_1,+}(q^{\frac{c}{2}} w) - \sum_{n \in \mathbb{Z}} \left(q^{-c} \frac{w}{z} \right)^n \Psi^{l_1,-}(q^{-\frac{c}{2}} w) \right) \quad (12)$$

and

$$E^{l_1, \pm}(z) E^{l_2, \pm}(w) = -E^{l_2, \pm}(w) E^{l_1, \pm}(z) \quad \text{for } a_{l_1, l_2} = 0$$

$$(wq^{\pm a_{l_1, l_2}} - z) E^{l_1, \pm}(z) E^{l_2, \pm}(w) = (zq^{\pm a_{l_1, l_2}} - w) E^{l_2, \pm}(w) E^{l_1, \pm}(z)$$

$$\left[[E^{2, \pm}(z), E^{1, \pm}(z')]_{q^{\pm 1}}, [E^{2, \pm}(w), E^{3, \pm}(w')]_{q^{\pm 1}} \right] + \left[[E^{2, \pm}(w), E^{1, \pm}(z')]_{q^{\pm 1}}, [E^{2, \pm}(z), E^{3, \pm}(w')]_{q^{\pm 1}} \right] = 0 \quad (13)$$

In (10)-(13), $j, j_1, j_2 = 1, 2, 3, 4$; $l, l_1, l_2 = 1, 2, 3$ and $a_{14} = a_{41} = a_{34} = a_{43} = 1$, $a_{24} = a_{42} = a_{44} = 0$. $U'_q(\widehat{gl}(2|2))$ supplemented by the grading operator d will be referred to as $U_q(\widehat{gl}(2|2))$. The operator d satisfies

$$w^{-d} E^{l, \pm}(z) w^d = w E^{l, \pm}(wz) \quad w^{-d} \Psi^{j, \pm}(z) w^d = \Psi^{j, \pm}(wz) \quad (14)$$

In terms of the basis $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ with the bilinear form $(\tau_i, \tau_j) = -(-1)^i \delta_{i, j}$, the classical roots $\bar{\alpha}_i$ are expressed by $\bar{\alpha}_i = -(-1)^i(\tau_i + \tau_{i+1})$ for $i = 1, 2, 3$ and $\bar{\alpha}_4 = \tau_1 - \tau_4$. The classical weights $\bar{\Lambda}_i$ read $\bar{\Lambda}_i = \sum_{j=1}^i \tau_j - \frac{1}{2} \sum_{j=1}^4 \tau_j$ with $i = 1, 2, 3, 4$. The affine root δ and the affine weight Λ_0 satisfy $(\Lambda_0, \Lambda_0) = (\delta, \delta) = (\tau_i, \Lambda_0) = (\tau_i, \delta) = 0$ and $(\Lambda_0, \delta) = 1$. Then the set of simple roots is given by $\alpha_0 = \delta - \bar{\alpha}_1 - \bar{\alpha}_2 - \bar{\alpha}_3$ and $\alpha_i = \bar{\alpha}_i$ for $i = 1, 2, 3, 4$. The free abelian group $P = \oplus_{i=0}^4 \mathbb{Z} \Lambda_i + \mathbb{Z} \delta$ with $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$ for $i = 1, 2, 3$ and $\Lambda_4 = \bar{\Lambda}_4$ is called the weight lattice. Via (\cdot, \cdot) , its dual lattice $P^* = \oplus_{i=0}^4 \mathbb{Z} h_i + \mathbb{Z} d$ can be identified with P by setting $\alpha_i = h_i$ and $d = \Lambda_0$.

Discarding the subset of generators $\Psi^{4, \pm}(z)$ from (10)-(13) yields a definition of $U'_q(\widehat{sl}(2|2))$. Chevalley and Drinfeld basis of $U'_q(\widehat{sl}(2|2))$ are related by

$$\begin{aligned} q^{\pm h_l} &= \Psi_0^{l, \pm} & q^{\pm h_0} &= q^{\pm(c-h_1-h_2-h_3)} \\ e_l &= E_0^{l, +} & f_l &= E_0^{l, -} & l &= 1, 2, 3 \\ e_0 &= [E_0^{3, -}, [E_0^{2, -}, E_1^{1, -}]_q]_q & & q^{-h_1-h_2-h_3} \\ f_0 &= q^{h_1+h_2+h_3} [[E_{-1}^{1, +}, E_0^{2, +}]_{q^{-1}}, E_0^{3, +}]_{q^{-1}} \end{aligned} \quad (15)$$

with the super brackets defined as

$$\begin{aligned} [E_{n''}^{l_3, \pm}, [E_{n'}^{l_2, \pm}, E_n^{l_1, \pm}]_p]_p &= -p^{a_{l_1, l_2} + a_{l_1, l_3} + a_{l_2, l_3}} [[E_n^{l_1, \pm}, E_{n'}^{l_2, \pm}]_{p^{-1}}, E_{n''}^{l_3, \pm}]_{p^{-1}} = \\ &= E_{n''}^{l_3, \pm} \left(E_{n'}^{l_2, \pm} E_n^{l_1, \pm} + p^{a_{l_1, l_2}} E_n^{l_1, \pm} E_{n'}^{l_2, \pm} \right) - p^{a_{l_1, l_3} + a_{l_2, l_3}} \left(E_{n'}^{l_2, \pm} E_n^{l_1, \pm} + p^{a_{l_1, l_2}} E_n^{l_1, \pm} E_{n'}^{l_2, \pm} \right) E_{n''}^{l_3, \pm} \end{aligned} \quad (16)$$

for $p = q$ or $p = q^{-1}$. $\Psi_0^{4, \pm}$ may be reintroduced as a Chevalley generator:

$$q^{\pm h_4} \equiv \Psi_0^{4, \pm} \quad (17)$$

The quantum superalgebras generated by $\{e_l, f_l, q^{\pm h_l}, l = 1, 2, 3\}$ and $\{e_l, f_l, q^{\pm h_j}, l = 1, 2, 3; j = 1, 2, 3, 4\}$ are denoted by $U_q(\widehat{sl}(2|2))$ and $U_q(\widehat{gl}(2|2))$, respectively. For several purposes, a reformulation of the generating functions $\Psi^{j, \pm}(z)$ in terms of $q^{\pm h_j}$ and generators H_n^j with $j = 1, 2, 3, 4$; $n \in \mathbb{Z} \setminus \{0\}$ proves useful:

$$\begin{aligned} \Psi^{j, +}(z) &= q^{h_j} \exp\left((q - q^{-1}) \sum_{n>0} H_n^j z^{-n}\right) \\ \Psi^{j, -}(z) &= q^{-h_j} \exp\left(-(q - q^{-1}) \sum_{n>0} H_{-n}^j z^n\right) \end{aligned} \quad (18)$$

A graded Hopf algebra structure for $U_q(\widehat{gl}(2|2))$ with the choice of simple roots adopted here is found in [42]. There the universal R-matrix is expressed in terms of the generators $\{c, E_n^{i, \pm}, H_{\pm n}^j, h_j, i = 1, 2, 3, j = 1, 2, 3, 4, n \in \mathbb{Z}, n' \in \mathbb{Z}_+\}$ and the grading operator d .

As is readily verified from (10) and (11), for each n the sum $H_n^1 + H_n^3$ commutes with any generator of $U'_q(\widehat{sl}(2|2))$. All combinations $H_n^1 + H_n^3$, $n \in \mathbb{Z} \setminus \{0\}$, commute among themselves and generate the commutative algebra \mathcal{H} . Therefore,

a representation of $U'_q(\widehat{sl}(2|2))$ can be realized as the tensor product of a one-dimensional representation of \mathcal{H} and a representation of $U'_q(\widehat{sl}(2|2))/\mathcal{H}$. In the present study, explicit reference will be made mainly to $U'_q(\widehat{sl}(2|2))$. For the discussion of weight spaces, the quantum super algebra $U_q(gl(2|2))$ is required as well. In this context it is useful to supplement the Chevalley basis (1), (2) of $U'_q(\widehat{sl}(2|2))$ by $q^{\pm h_4}$. Then the first line of (1) applies to $i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2, 3$ with $a_{04} = a_{40} = -2$.

If c acts as a scalar on a module of the quantum affine superalgebra, then this scalar is referred to as the level of the module. For the affine Lie superalgebra $\widehat{gl}(m|m')$, the irreducible, nonzero level modules with finite-dimensional weight spaces are classified in [32]. Since the notion of integrability formulated for affine algebras [33] proves too restrictive for affine Lie superalgebras [34], the concept of weak integrability has been introduced [35]. The affine Lie superalgebra \hat{g} is associated to the Lie superalgebra $g = g_{\bar{0}} + g_{\bar{1}}$ with an even symmetric bilinear form $(\cdot|\cdot)$. Even means that $(g_{\bar{0}}|g_{\bar{1}}) = 0$ and symmetric indicates that $(\cdot|\cdot)$ is symmetric on $g_{\bar{0}}$ and skewsymmetric on $g_{\bar{1}}$ [36]. Furthermore, $g_{\bar{0}}$ is taken reductive, i.e. $g_{\bar{0}} = \bigoplus_{j=0}^J g_{\bar{0},j}$, where $g_{\bar{0},0}$ is abelian and $g_{\bar{0},j}$ with $j > 0$ are simple Lie algebras. Given a subset $J' \subset \{1, 2, \dots, J\}$, a \hat{g} -module M is called weakly integrable (or J' -integrable), if

1. the Cartan subalgebra \hat{h} of \hat{g} is diagonalizable on M
2. G is locally finite on M
3. M is integrable as a $\hat{g}_{\bar{0},j}$ -module for all $j \in J'$

Here $\hat{g}_{\bar{0},j}$ denotes the affine Lie algebra associated to $g_{\bar{0},j}$. As shown in [32], any irreducible weakly integrable module is a highest weight module. In [35], all irreducible weakly integrable modules with level one have been classified for $\widehat{gl}(m|m')$. To each $s \in \mathbb{Z}$ there corresponds exactly one such module. Character formulae of various types are available for these modules [35], [37].

For the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$, a free boson realization at level one has been employed in [43] to classify infinitely many highest weight vectors. Based on an investigation of the Fock spaces by means of the technique outlined in [44], character formulae have been conjectured for the corresponding level-one modules of $U_q(\widehat{sl}(2|1))$. This type of analysis has also been applied to $U_q(\widehat{gl}(2|2))$ in [45]. The classifications and character formulae in [32], [35] and [43] refer to the standard set of simple roots for $sl(m|m')$ containing only one unique odd simple root. In contrast, [45] employs the simple root set characterized by (3). It turns out that all but one irreducible level-one modules relevant to the model studied below are found among the (infinitely many) modules listed in [45]. A convenient tool to transform between the different simple root systems and the related modules is provided by the application of odd reflections [46,35].

Starting with [38], the representation theory of $sl(m|m')$ or $gl(m|m')$ has been widely studied. References can be found in [34] or [39], [40], for example.

A four-dimensional module W of $U'_q(\widehat{sl}(2|2))$ with basis $\{w_j\}_{0 \leq j \leq 3}$ is provided by

$$\begin{aligned} h_k w_j &= (-1)^j (\delta_{j,k-1} - \delta_{j,k}) w_j & h_0 w_j &= -(h_1 + h_2 + h_3) w_j \\ f_k w_{k-1} &= w_k & f_0 w_3 &= -q^{-1} w_0 \\ e_k w_k &= (-1)^{k-1} w_{k-1} & e_0 w_0 &= q w_3 \end{aligned} \tag{19}$$

with $k = 1, 2, 3$; $j = 0, 1, 2, 3$. Furthermore, W^* is introduced as the dual space of W endowed with the $U'_q(\widehat{sl}(2|2))$ - (or $U'_q(\widehat{gl}(2|2))$)-structure

$$\langle aw^* | w \rangle = (-1)^{|a| \cdot |w^*|} \langle w^* | S(a)w \rangle \quad \forall a \in U'_q(\widehat{sl}(2|2)) \quad \left(\text{or } a \in U'_q(\widehat{gl}(2|2)) \right) \tag{20}$$

Its basis $\{w_j^*\}_{0 \leq j \leq 3}$ is chosen such that the action of $U'_q(\widehat{sl}(2|2))$ reads

$$\begin{aligned} h_k w_j^* &= (-1)^j (\delta_{j,k} - \delta_{j,k-1}) w_j^* & h_0 w_j^* &= -(h_1 + h_2 + h_3) w_j^* \\ f_k w_k^* &= (-1)^{k-1} q^{(-1)^{k-1}} w_{k-1}^* & f_0 w_0^* &= q^{-2} w_3^* \\ e_k w_{k-1}^* &= -q^{(-1)^k} w_k^* & e_0 w_3^* &= q^2 w_0^* \end{aligned} \tag{21}$$

where $k = 1, 2, 3$; $j = 0, 1, 2, 3$. The \mathbb{Z}_2 -grading on W and W^* is fixed by $|w_j| = |w_j^*| = \frac{1}{2}(1 - (-1)^j)$. A $U_q(gl(2|2))$ -structure on W and W^* is obtained from the left columns of (19) and (21) and

$$h_4 w_j = (\delta_{j,0} - \delta_{j,3}) w_j \quad h_4 w_j^* = (\delta_{j,3} - \delta_{j,0}) w_j^* \quad (22)$$

On the evaluation modules $W_z = W \otimes \mathbb{C}[z, z^{-1}]$ and $W_z^* = W^* \otimes \mathbb{C}[z, z^{-1}]$ a $U_q(\widehat{sl}(2|2))$ -structure can be introduced via

$$\begin{aligned} e_k(v_j \otimes z^n) &= e_k v_j \otimes z^{n+\delta_{k,0}} & f_k(v_j \otimes z^n) &= f_k v_j \otimes z^{n-\delta_{k,0}} \\ h_k(v_j \otimes z^n) &= h_k v_j \otimes z^n & d(v_j \otimes z^n) &= n v_j \otimes z^n \end{aligned} \quad (23)$$

for $j, k = 0, 1, 2, 3$ and $v_j = w_j$ or $v_j = w_j^*$. Throughout the paper, the structure (23) will be called the level-zero action of $U_q(\widehat{sl}(2|2))$. In contrast, the q -analogies of Yangian actions defined on irreducible level-one modules of $U_q(\widehat{sl}(N))$ are referred to as level-zero action in references [22, 24–29].

For the tensor product of two evaluation modules $V_{z_1}^{(1)}$ and $V_{z_2}^{(2)}$, the R-matrix $R(z_1/z_2) \in \text{End}(V_{z_1}^{(1)} \otimes V_{z_2}^{(2)})$ intertwines the action of $U_q(\widehat{sl}(2|2))$ (or $U_q(\widehat{gl}(2|2))$):

$$R(z_1/z_2) \Delta(a) = \Delta'(a) R(z_1/z_2) \quad \forall a \in U_q(\widehat{sl}(2|2)) \text{ (or } U_q(\widehat{gl}(2|2))) \quad (24)$$

where $\Delta' = \sigma \circ \Delta$ and $\sigma(a \otimes b) = (-1)^{|a| \cdot |b|} b \otimes a$. In the following, the choice of evaluation modules will be indicated by subscripts. The construction of the vertex model involves the matrix elements of $R_{WW}(z)$, $R_{WW^*}(z)$, $R_{W^*W}(z)$ and $R_{W^*W^*}(z)$ introduced by

$$\begin{aligned} R_{WW}(z_1/z_2)(w_{j_1} \otimes w_{k_1}) &= \sum_{j_2, k_2=0}^3 R_{j_1, k_1}^{j_2, k_2}(z_1/z_2) w_{j_2} \otimes w_{k_2} \\ R_{WW^*}(z_1/z_2)(w_{j_1} \otimes w_{k_1}^*) &= \sum_{j_2, k_2=0}^3 R_{j_1, k_1}^{j_2, k_2^*}(z_1/z_2) w_{j_2} \otimes w_{k_2}^* \\ R_{W^*W}(z_1/z_2)(w_{j_1}^* \otimes w_{j_1}) &= \sum_{j_2, k_2=0}^3 R_{j_1^*, k_1}^{j_2^*, k_2}(z_1/z_2) w_{j_2}^* \otimes w_{k_2} \\ R_{W^*W^*}(z_1/z_2)(w_{j_1}^* \otimes w_{k_1}^*) &= \sum_{j_2, k_2=0}^3 R_{j_1^*, k_1^*}^{j_2^*, k_2^*}(z_1/z_2) w_{j_2}^* \otimes w_{k_2}^* \end{aligned} \quad (25)$$

Up to a scalar multiple, the solutions of the intertwining condition (24) on $W_{z_1} \otimes W_{z_2}$ and on $W_{z_1} \otimes W_{z_2}^*$ with $z = z_1/z_2$ are given by

$$\begin{aligned} R_{0,0}^{0,0}(z) &= R_{2,2}^{2,2}(z) = 1 & R_{1,1}^{1,1}(z) &= R_{3,3}^{3,3}(z) = \frac{q^2 - z}{1 - q^2 z} \\ R_{j,k}^{j,k}(z) &= \frac{q(1-z)}{1 - q^2 z} & j &\neq k \\ R_{j,k}^{k,j}(z) &= -(-1)^{|j| \cdot |k|} \frac{z(q^2 - 1)}{1 - q^2 z} & j &< k \\ R_{j,k}^{k,j}(z) &= -(-1)^{|j| \cdot |k|} \frac{q^2 - 1}{1 - q^2 z} & j &> k \end{aligned} \quad (26)$$

and

$$\begin{aligned}
R_{0,0^*}^{0,0^*}(z) &= R_{2,2^*}^{2,2^*}(z) = 1 & R_{1,1^*}^{1,1^*}(z) &= R_{3,3^*}^{3,3^*}(z) = \frac{1-q^2z}{q^2-z} \\
R_{j,k^*}^{j,k^*}(z) &= \frac{q(1-z)}{q^2-z} & j &\neq k \\
R_{j,j^*}^{k,k^*}(z) &= (-1)^{|j|} \frac{q^2-1}{q^2-z} & j &< k \\
R_{j,j^*}^{k,k^*}(z) &= (-1)^{|j|} \frac{z(q^2-1)}{q^2-z} & j &> k
\end{aligned} \tag{27}$$

where $j, k = 0, 1, 2, 3$ and $|j| \equiv \frac{1}{2}(1 - (-1)^j)$. All other matrix elements of $R_{WW}(z)$ and $R_{WW^*}(z)$ vanish. The matrix elements of $R_{W^*W}(z)$ and $R_{W^*W^*}(z)$ are obtained from (26) and (27) via

$$\begin{aligned}
R_{j_1^*, j_2^*}^{k_1^*, k_2^*}(z) &= (-1)^{|j_1|+|k_1|} R_{k_1, k_2}^{j_1, j_2}(z) \\
R_{j_1^*, j_2^*}^{k_1^*, k_2^*}(z) &= R_{k_1, k_2}^{j_1, j_2}(z)
\end{aligned} \tag{28}$$

The R-matrix elements (26)-(28) yield Boltzmann weights for an integrable vertex model with the modules W and W^* attributed alternately to both its horizontal and vertical lines (see Fig. 1). An analogous model has been investigated for the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$ in [12]. For further analysis, four neighboring vertices of different type are collected in composite vertices of type A or B as illustrated in Fig. 2. A Boltzmann weight depending on the configuration of basis elements $\{w_j\}_{0 \leq j \leq 3}$ or $\{w_j^*\}_{0 \leq j \leq 3}$ on the outer links as well as on the spectral parameters assigned to the elementary vertices is provided by

$$\begin{aligned}
R_{j_1^*, j_2^*; j_3, j_4}^{k_1^*, k_2^*; k_3, k_4}(w, z) &= \sum_{l_1, l_2, l_3, l_4=0}^3 \bar{R}_{l_1, l_4}^{k_2^*, k_3}(w^{-1}z) \bar{R}_{l_3, l_4}^{l_4^*, k_4^*}(z) \bar{R}_{j_1, l_2}^{k_1, l_1}(z) \bar{R}_{j_2, j_3}^{l_2^*, l_3^*}(wz) & \text{type } A \\
R_{j_1^*, j_2^*; j_3^*, j_4^*}(w, z) &= \sum_{l_1, l_2, l_3, l_4=0}^3 \bar{R}_{l_1^*, l_4^*}^{k_2^*, k_3^*}(wz) \bar{R}_{j_1^*, l_2^*}^{k_1^*, l_1^*}(z) \bar{R}_{l_3, j_4}^{l_4, k_4}(z) \bar{R}_{j_2, j_3^*}^{l_2^*, l_3^*}(w^{-1}z) & \text{type } B
\end{aligned} \tag{29}$$

In (29), $\bar{R}(z) = \sigma \cdot R(z)$. The assignment of indices and spectral parameters is shown in Fig. 2. As a consequence of the initial condition $\bar{R}_{j_1, j_2}^{k_1, k_2}(1) = \bar{R}_{j_1^*, j_2^*}^{k_1^*, k_2^*}(1) = \delta_{j_1, k_1} \delta_{j_2, k_2}$ and the unitarity relation

$$\sum_{l_1, l_2=0}^3 \bar{R}_{l_1, l_2}^{k_1^*, k_2}(w^{-1}) \bar{R}_{j_1^*, j_2}^{l_1^*, l_2^*}(w) = \delta_{j_1, k_1} \delta_{j_2, k_2} \cdot \frac{q^2(1-w)^2}{(q^2-w)(1-q^2w)} \tag{30}$$

the composite R-matrices at $z = 1$ act as a scalar multiples on $W \otimes W^* \otimes W \otimes W^*$ or $W^* \otimes W \otimes W^* \otimes W$:

$$\begin{aligned}
R_{j_1^*, j_2^*; j_3, j_4}^{k_1^*, k_2^*; k_3, k_4}(w, 1) &= \delta_{j_1, k_1} \delta_{j_2, k_2} \delta_{j_3, k_3} \delta_{j_4, k_4} \cdot \frac{q^2(1-w)^2}{(q^2-w)(1-q^2w)} \\
R_{j_1^*, j_2^*; j_3^*, j_4^*}(w, 1) &= \delta_{j_1, k_1} \delta_{j_2, k_2} \delta_{j_3, k_3} \delta_{j_4, k_4} \cdot \frac{q^2(1-w)^2}{(q^2-w)(1-q^2w)}
\end{aligned} \tag{31}$$

Equation (31) states an initial condition for the composite R-matrices.

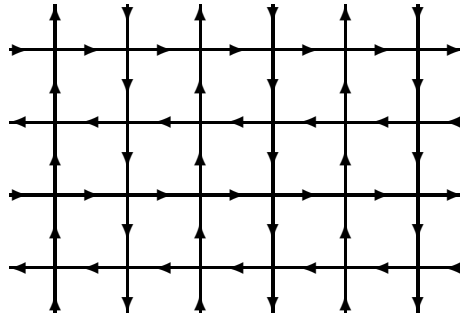


Fig. 1: The lattice model

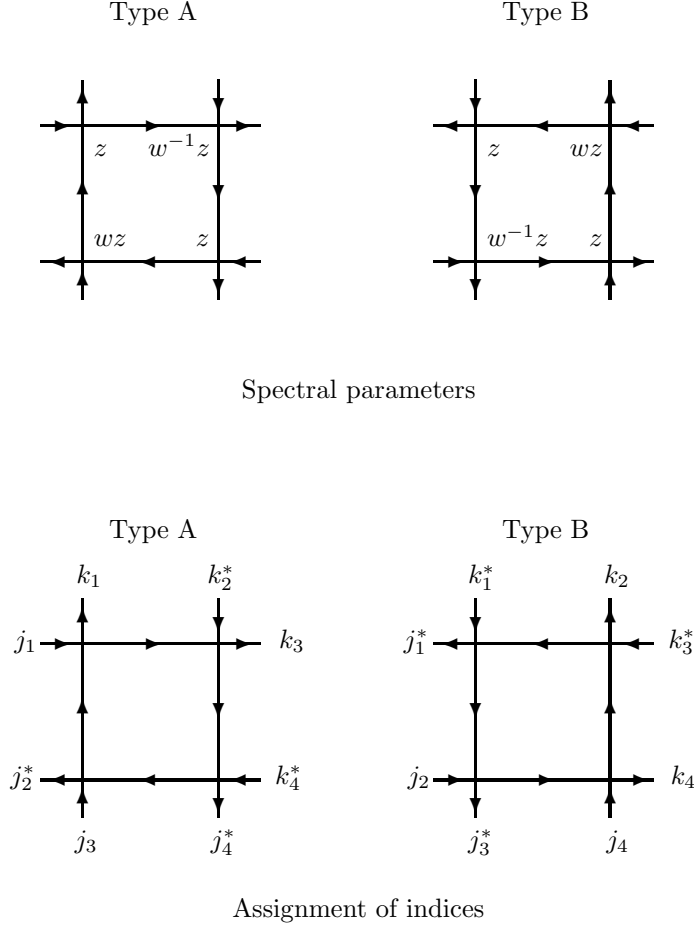


Fig.2: The composite vertices

III. CORNER TRANSFER MATRIX IN THE LIMIT $Q \rightarrow 0$

For vertex models based on quantum affine algebras, the space of states has been interpreted in the framework of the corresponding representation theory [5,2]. The analysis relies on the spectrum of the corner transfer matrix Hamiltonian [1] acting on half-infinite tensor products of evaluation modules subject to a boundary condition. Quite generally, the key to its evaluation is the limit of vanishing deformation parameter q . This section considers the same limit for the inhomogeneous vertex model specified in the previous section. First a lattice decomposed into composite vertices of type A with the corresponding R-matrix denoted by $R(w, z)$ will be examined.

The limits of each R-matrix element (26)-(28) are well defined. However, in the limit $q \rightarrow 0$, the composite R-matrix $R(w, z)$ does not give rise to an invertible map $(W \otimes W^*)^{\otimes 2} \rightarrow (W \otimes W^*)^{\otimes 2}$. An invertible map emerges taking the limit $q \rightarrow 0$ of $R(q^2 w, z)$. While some single matrix elements on $W \otimes W^*$ or $W^* \otimes W$ diverge, the composite R-matrix $R(q^2 w, z)$ always remains well-defined. This observation applies to the analogous construction for the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$ as well [12]. To define the corner transfer matrix Hamiltonian, consider the triangular section A_N of the lattice model introduced in Sect. II with a vertical (horizontal) boundary formed by $2N + 1$ horizontal (vertical) links on the boundaries of the upper left quadrant. A Hamiltonian $h_{CTM}^{(N,q)}(w) : W^* \otimes (W \otimes W^*)^{\otimes (N+1)} \rightarrow W^* \otimes (W \otimes W^*)^{\otimes (N+1)}$ associated to the section A_N is introduced by

$$h_{CTM}^{(N,q)}(w) = (2N+3)\tilde{h}_{2N+3,2N+2} + \sum_{\hat{N}=1}^N \hat{N} h_{2\hat{N}+2,2\hat{N}+1,2\hat{N},2\hat{N}-1}(w) \quad (32)$$

In (32), $\tilde{h}_{2N+3,2N+2}$ denotes the operator \tilde{h} obtained from $\bar{R}_{W^*W^*}(z) = 1 + (z-1)\tilde{h} + O((z-1)^2)$ acting on the two leftmost components of $W^* \otimes (W \otimes W^*)^{\otimes(N+1)}$ and $h_{2\hat{N}+2,2\hat{N}+1,2\hat{N},2\hat{N}-1}(w) = 1^{\otimes(2N-2\hat{N}+1)} \otimes h^{(q)}(w) \otimes 1^{\otimes(2\hat{N}-2)}$ is defined by the expansion

$$\frac{(1-wz)(z-q^4w)}{(1-q^2wz)(z-q^2w)} R(q^2w, z) = 1 + (z-1)h^{(q)}(w) + O((z-1)^2) \quad (33)$$

The prefactor on the lhs takes into account the scalar function in the unitarity relation (30). From (32), the corner transfer matrix Hamiltonian is obtained by performing the limit $N \rightarrow \infty$ with respect to a suitable boundary condition. To facilitate the limit $q \rightarrow 0$, $R(q^2w, z)$ may be decomposed according to

$$R_{j_1, j_2^*; j_3, j_4^*}^{k_1, k_2^*; k_3, k_4^*}(q^2w, z) = \frac{(1-w)(1-q^4w)}{(1-q^2w)^2} \sum_{l_1, l_2=0}^3 Y_{j_1, l_2^*, l_1}^{k_1, k_2^*, k_3}(w, w^{-1}z) X_{j_2^*, j_3, j_4^*}^{l_1^*, l_1, k_4^*}(wz, w^{-1}) \quad (34)$$

with

$$\begin{aligned} X_{l_1^*, l_2, j_4^*}^{k_2^*, k_3, k_4^*}(w_1, w_2) &\equiv \sum_{l_1, l_2, l_3=0}^3 \bar{R}_{l_3, l_5^*}^{k_2^*, k_3}(q^{-2}w_2) \bar{R}_{l_4^*, j_4^*}^{l_5^*, k_4^*}(w_1 w_2) \bar{R}_{l_1^*, l_2}^{l_3, l_4^*}(q^2w_1) \\ Y_{j_1, j_2^*, j_3}^{k_1, l_1^*, l_2}(w_1, w_2) &\equiv \sum_{l_1, l_2, l_3=0}^3 \bar{R}_{l_5^*, l_4^*}^{l_1^*, l_2}(q^{-2}w_2) \bar{R}_{j_1, l_3}^{k_1, l_5}(w_1 w_2) \bar{R}_{j_2^*, j_3}^{l_3, l_4^*}(q^2w_1) \end{aligned} \quad (35)$$

Equation (34) implies a decomposition of $h^{(q)}(w)$ introduced in (33):

$$h^{(q)}(w) = h_X(w) + h_Y(w) \quad (36)$$

where

$$\begin{aligned} \frac{(1-q^4w)(1-wz)}{(1-q^2w)(1-q^2wz)} X(wz, w^{-1}) &= 1 + (z-1)h_X(w) + O((z-1)^2) \\ \frac{(1-w)(q^4w-z)}{(1-q^2w)(q^2w-z)} Y(w, w^{-1}z) &= 1 + (z-1)h_Y(w) + O((z-1)^2) \end{aligned} \quad (37)$$

For convenience, the limits of the matrix elements of $X(w_1, w_2)$ and $Y(w_1, w_2)$ may be denoted by

$$\begin{aligned} x_{j_1^*, j_2^*, j_3}^{k_1^*, k_2^*, k_3^*}(w_1, w_2) &\equiv \lim_{q \rightarrow 0} X_{j_1^*, j_2^*, j_3}^{k_1^*, k_2^*, k_3^*}(w_1, w_2) \\ y_{j_1, j_2^*, j_3}^{k_1, k_2^*, k_3}(w_1, w_2) &\equiv \lim_{q \rightarrow 0} Y_{j_1, j_2^*, j_3}^{k_1, k_2^*, k_3}(w_1, w_2) \end{aligned} \quad (38)$$

Insertion of equations (26)-(28) in (35) yields

$$\begin{aligned} x_{j_1^*, j_2^*, j_3}^{j_1^*, j_2^*, j_3^*}(wz, w^{-1}) &= y_{j_3^*, j_2^*, j_1^*}^{j_3^*, j_2^*, j_1^*}(w, w^{-1}z) = \frac{z}{1-w} \quad \text{if } j_1 > j_3 \text{ or } j_1 = j_3 = 1, 3 \\ x_{j_1^*, j_2^*, j_3}^{j_1^*, j_2^*, j_3^*}(wz, w^{-1}) &= y_{j_3^*, j_2^*, j_1^*}^{j_3^*, j_2^*, j_1^*}(w, w^{-1}z) = \frac{1}{1-w} \quad \text{if } j_1 < j_3 \text{ or } j_1 = j_3 = 0, 2 \end{aligned} \quad (39)$$

and

$$x_{j_1^*, j_1, j_3}^{j_3^*, j_2^*, j_2^*}(wz, w^{-1}) = y_{j_3, j_1^*, j_1}^{j_2^*, j_2^*, j_3}(w, w^{-1}z) = 0 \quad \text{if } j_1 > j_2 \text{ or } 0, 2 = j_2 = j_1 \neq j_3$$

$$x_{j_1^*, j_1, j_3^*}^{j_3^*, j_2, j_2^*}(wz, w^{-1}) = y_{j_3, j_1^*, j_1}^{j_2, j_2^*, j_3}(w, w^{-1}z) = \frac{1-z}{1-w} \quad \text{if } j_1 < j_2 \text{ or } 1, 3 = j_2 = j_1 \neq j_3 \quad (40)$$

and

$$x_{j_3^*, j_1, j_1^*}^{j_2^*, j_2, j_3^*}(wz, w^{-1}) = y_{j_1, j_1^*, j_3}^{j_3, j_2^*, j_2}(w, w^{-1}z) = -(-1)^{|j_1|+|j_2|} \frac{1-z}{1-w} \quad \text{if } j_1 < j_2 \text{ or } 0, 2 = j_2 = j_1 \neq j_3$$

$$x_{j_3^*, j_1, j_1^*}^{j_2^*, j_2, j_3^*}(wz, w^{-1}) = y_{j_1, j_1^*, j_3}^{j_3, j_2^*, j_2}(w, w^{-1}z) = 0 \quad \text{if } j_1 > j_2 \text{ or } 1, 3 = j_2 = j_1 \neq j_3 \quad (41)$$

Clearly, in the limit $q \rightarrow 0$, $h^{(q)}(w)$ and $h_{CTM}^{(N,q)}(w)$ become independent of w . Hence they may be denoted by

$$h \equiv \lim_{q \rightarrow 0} h^{(q)}(w) \quad (42)$$

and

$$h_{CTM}^{(N)} \equiv \lim_{q \rightarrow 0} h_{CTM}^{(N,q)}(w) = \lim_{q \rightarrow 0} \left((2N+3) \tilde{h}_{2N+3, 2N+2} + \sum_{\hat{N}=1}^N \hat{N} h_{2\hat{N}+2, 2\hat{N}+1, 2\hat{N}, 2\hat{N}-1}(w) \right) \quad (43)$$

With (39)-(41), it is easily verified that the matrix elements of $h_{CTM}^{(N)}$ form a triangular matrix. A particular configuration $(w_{j_{2N+3}}^* \otimes \dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ may be abbreviated by $(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*)$. The matrix element $h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*)}$ of $h_{CTM}^{(N)}$ is defined by

$$h_{CTM}^{(N)}(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*) = \sum_{k_r=0}^3 h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*)} (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*) \quad (44)$$

Suppose $h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*)} \neq 0$. Then either

$$(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*) = (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*) \quad \text{or} \quad \sum_{r=1}^{2N+3} j_r < \sum_{r=1}^{2N+3} k_r \quad (45)$$

or $(k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*)$ results from $(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*)$ by one of the following eight replacements:

$$\begin{array}{llll} (j^*, 0, 0^*) \longrightarrow (0^*, 0, j^*) & (0, 0^*, j) \longrightarrow (j, 0^*, 0) & j \neq 0 \\ (j^*, 2, 2^*) \longrightarrow (2^*, 2, j^*) & (2, 2^*, j) \longrightarrow (j, 2^*, 2) & j \neq 2 \\ (j, 1^*, 1) \longrightarrow (1, 1^*, j) & (1^*, 1, j^*) \longrightarrow (j^*, 1, 1^*) & j \neq 1 \\ (j, 3^*, 3) \longrightarrow (3, 3^*, j) & (3^*, 3, j^*) \longrightarrow (j^*, 3, 3^*) & j \neq 3 \end{array} \quad (46)$$

Thus the matrix $h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (k_{2N+3}^*, \dots, k_4, k_3^*, k_2, k_1^*)}$ is triangular. In particular, $(3^*, \dots, 3, 3^*, 3, 3^*)$ is an eigenvector of $h_{CTM}^{(N)}$. According to (39), the diagonal elements $h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*)}$ decouple into a contribution depending only on the entries $(j_{2N+3}^*, j_{2N+1}^*, \dots, j_3^*, j_1^*)$ and into a second one depending only on the remaining entries $(j_{2N+2}, j_{2N}, \dots, j_4, j_2)$. The diagonal elements are restricted by

$$h_{(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*); (j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*)} \leq h_{(3^*, 3, 3^*, 3, \dots, 3^*, 3, 3^*); (3^*, 3, 3^*, 3, \dots, 3^*, 3, 3^*)} = 2 \quad (47)$$

for any $(j_{2N+3}^*, \dots, j_4, j_3^*, j_2, j_1^*)$. Regarding $(\dots \otimes w_3 \otimes w_3^* \otimes w_3 \otimes w_3^*)$ as a ground state configuration, a CTM-Hamiltonian at $q = 0$ may be defined on the set Ω_A of all configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ in the half-infinite tensor product $\dots \otimes W \otimes W^* \otimes W \otimes W^*$ with $j_{2r} = 3$ and $j_{2r-1} = 3^*$ for almost all r . With respect to this restricted set, the limit $N \rightarrow \infty$ can be performed for (43) after replacing $\frac{1}{2}h^{(q)}(w)$ by $\frac{1}{2}h^{(q)}(w) - 1$ in the definition of $h_{2\hat{N}+2, 2\hat{N}+1, 2\hat{N}, 2\hat{N}-1}(w)$ and discarding the part containing $\lim_{q \rightarrow 0} \tilde{h}$. This amounts to rescaling the matrix $R(q^2w, z)$ by a factor z^{-2} which does not affect integrability. Renormalizations of that type are familiar from the vertex models related to quantum affine algebras [3]. The diagonal elements $h_{(\dots, j_4, j_3^*, j_2, j_1^*); (\dots, j_4, j_3^*, j_2, j_1^*)}^{ren}$ of the resulting CTM-Hamiltonian h_{CTM} are obtained from (36), (37) and (39):

$$h_{(\dots, j_4, j_3^*, j_2, j_1^*), (\dots, j_4, j_3^*, j_2, j_1^*)}^{ren} = - \sum_{r=1}^{\infty} r (x_{j_{2r+1}, j_{2r-1}} + y_{j_{2r+2}, j_{2r}}) \quad (48)$$

with

$$x_{j_1, j_2} = y_{j_2, j_1} = \begin{cases} 0 & j_1 > j_2 \quad \text{or} \quad j_1 = j_2 = 1, 3 \\ 1 & j_1 < j_2 \quad \text{or} \quad j_1 = j_2 = 0, 2 \end{cases} \quad (49)$$

The decomposition of the diagonal elements (48) into two parts depending either only on the even or on the odd components of the tensor product $(\dots \otimes W \otimes W^* \otimes W \otimes W^*)$ suggests to classify the configurations $(\dots \otimes w_{j_5}^* \otimes w_{j_3}^* \otimes w_{j_1}^*)$ and $(\dots \otimes w_{j_6} \otimes w_{j_4} \otimes w_{j_2})$ according to the action of $x_{j_{2r+1}, j_{2r-1}}$ and $y_{j_{2r+2}, j_{2r}}$ on $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*) \in \Omega_A$, respectively. Writing $(\dots, j_5^*, j_3^*, j_1^*) \equiv (\dots \otimes w_{j_5}^* \otimes w_{j_3}^* \otimes w_{j_1}^*)$ and $(\dots, j_6, j_4, j_2) \equiv (\dots \otimes w_{j_6} \otimes w_{j_4} \otimes w_{j_2})$, the sets corresponding to the contribution $-n$ may be denoted by

$$\begin{aligned} \{\tau^*\}_{-n} &= \left\{ (\dots, j_5^*, j_3^*, j_1^*) \in (\dots \otimes W^* \otimes W^* \otimes W^*) \mid j_r^* = 3^* \forall r > R \in \mathbb{N} \text{ and } \sum_{r=1}^{\infty} r x_{j_{2r+1}, j_{2r-1}} = n \right\} \\ \{\tau\}_{-n} &= \left\{ (\dots, j_6, j_4, j_2) \in (\dots \otimes W \otimes W \otimes W) \mid j_r = 3 \forall r > R \in \mathbb{N} \text{ and } \sum_{r=1}^{\infty} r y_{j_{2r+2}, j_{2r}} = n \right\} \end{aligned} \quad (50)$$

with $n = 0, 1, 2, \dots$. Due to (49),

$$\begin{aligned} \{\tau\}_0 &= \{(\dots, 3, 3, 3)\} \\ \{\tau\}_{-1} &= \{(\dots, 3, 3, 0), (\dots, 3, 3, 1), (\dots, 3, 3, 2)\} \\ \{\tau\}_{-2} &= \{(\dots, 3, 3, 0, 1), (\dots, 3, 3, 0, 2), (\dots, 3, 3, 0, 3), (\dots, 3, 3, 1, 1), (\dots, 3, 3, 1, 2), (\dots, 3, 3, 1, 3), \\ &\quad (\dots, 3, 3, 2, 3)\} \\ \{\tau\}_{-3} &= \{(\dots, 3, 3, 0, 1, 1), (\dots, 3, 3, 0, 1, 2), (\dots, 3, 3, 0, 1, 3), (\dots, 3, 3, 0, 2, 3), (\dots, 3, 3, 0, 3, 3), \\ &\quad (\dots, 3, 3, 1, 1, 1), (\dots, 3, 3, 1, 1, 2), (\dots, 3, 3, 1, 1, 3), (\dots, 3, 3, 1, 2, 3), (\dots, 3, 3, 1, 3, 3), \\ &\quad (\dots, 3, 3, 2, 3, 3), \\ &\quad (\dots, 3, 3, 0, 0), (\dots, 3, 3, 1, 0), (\dots, 3, 3, 2, 0), (\dots, 3, 3, 2, 1), (\dots, 3, 3, 2, 2)\} \end{aligned} \quad (51)$$

In contrast to the sets $\{\tau\}_{-n}$, each set $\{\tau^*\}_{-n}$ contains infinitely many configurations. For example,

$$\begin{aligned} \{\tau^*\}_0 &= \{(\dots, 3^*, 3^*, 3^*), (\dots, 3^*, 3^*, 2^*), (\dots, 3^*, 3^*, 0^*), (\dots, 3^*, 3^*, 2^*, 0^*), \\ &\quad (\dots, 3^*, 3^*, (1^*)^m), (\dots, 3^*, 3^*, 2^*, (1^*)^m), (\dots, 3^*, 3^*, (1^*)^m, 0^*), (\dots, 3^*, 3^*, 2^*, (1^*)^m, 0^*)\}_{m \in \mathbb{N}} \end{aligned} \quad (52)$$

In general, the set $\{\tau^*\}_{-n}$ is more conveniently specified in terms of finitely many subsets of configurations. For this purpose, two types of subsets $\{\Theta_{1^*}, j_{2R'-1}^*, \dots, j_3^*, j_1^*\}$ and $\{\Theta_{3^*}, j_{2R'-1}^*, \dots, j_3^*, j_1^*\}$ with finite R' are introduced:

$$\begin{aligned} \{\Theta_{1^*}, j_{2R'-1}^*, \dots, j_3^*, j_1^*\} &\equiv \left\{ (\dots, 3^*, 3^*, 2^*, j_{2R'-1}^*, \dots, j_3^*, j_1^*), (\dots, 3^*, 3^*, (1^*)^m, j_{2R'-1}^*, \dots, j_3^*, j_1^*), \right. \\ &\quad \left. (\dots, 3^*, 3^*, 2^*, (1^*)^m, j_{2R'-1}^*, \dots, j_3^*, j_1^*) \right\}_{m \in \mathbb{N}} \end{aligned}$$

$$\begin{aligned} \{\Theta_{3^*}, j_{2R'-1}^*, \dots, j_3^*, j_1^*\} \equiv & \left\{ (\dots, 3^*, 3^*, 0^*, j_{2R'-1}^*, \dots, j_3^*, j_1^*), (\dots, 3^*, 3^*, 2^*, 0^*, j_{2R'-1}^*, \dots, j_3^*, j_1^*), \right. \\ & \left. (\dots, 3^*, 3^*, (1^*)^m, 0^*, j_{2R'-1}^*, \dots, j_3^*, j_1^*), (\dots, 3^*, 3^*, 2^*, (1^*)^m, 0^*, j_{2R'-1}^*, \dots, j_3^*, j_1^*) \right\}_{m \in \mathbb{N}} \end{aligned} \quad (53)$$

Then the three sets $\{\tau^*\}_{-n}$ with $n = 1, 2, 3$ read

$$\begin{aligned} \{\tau^*\}_{-1} = & \left\{ \{\Theta_{1^*}, 2^*\}, \{\Theta_{1^*}, 3^*\}, \{\Theta_{3^*}, 0^*\}, \{\Theta_{3^*}, 1^*\}, \{\Theta_{3^*}, 2^*\}, \{\Theta_{3^*}, 3^*\} \right\} \\ \{\tau^*\}_{-2} = & \left\{ \{\Theta_{1^*}, 3^*, 0^*\}, \{\Theta_{1^*}, 3^*, 1^*\}, \{\Theta_{1^*}, 3^*, 2^*\}, \{\Theta_{1^*}, 3^*, 3^*\}, \{\Theta_{1^*}, 2^*, 0^*\}, \{\Theta_{1^*}, 2^*, 1^*\}, \right. \\ & \{\Theta_{3^*}, 3^*, 0^*\}, \{\Theta_{3^*}, 3^*, 1^*\}, \{\Theta_{3^*}, 3^*, 2^*\}, \{\Theta_{3^*}, 3^*, 3^*\}, \\ & \left. \{\Theta_{3^*}, 2^*, 0^*\}, \{\Theta_{3^*}, 2^*, 1^*\}, \{\Theta_{3^*}, 1^*, 0^*\}, \{\Theta_{3^*}, 1^*, 1^*\} \right\} \\ \{\tau^*\}_{-3} = & \left\{ \{\Theta_{1^*}, 2^*, 3^*\}, \{\Theta_{1^*}, 2^*, 2^*\}, \{\Theta_{1^*}, 3^*, 3^*, k^*\}, \{\Theta_{1^*}, 3^*, 2^*, 0^*\}, \{\Theta_{1^*}, 3^*, 2^*, 1^*\}, \right. \\ & \{\Theta_{1^*}, 3^*, 1^*, 0^*\}, \{\Theta_{1^*}, 3^*, 1^*, 1^*\}, \{\Theta_{1^*}, 2^*, 1^*, 0^*\}, \{\Theta_{1^*}, 2^*, 1^*, 1^*\}, \\ & \{\Theta_{3^*}, 2^*, 2^*\}, \{\Theta_{3^*}, 2^*, 3^*\}, \{\Theta_{3^*}, 1^*, 2^*\}, \{\Theta_{3^*}, 1^*, 3^*\}, \{\Theta_{3^*}, 0^*, k^*\}, \{\Theta_{3^*}, 3^*, 3^*, k^*\}, \\ & \{\Theta_{3^*}, 3^*, 2^*, 0^*\}, \{\Theta_{3^*}, 3^*, 2^*, 1^*\}, \{\Theta_{3^*}, 3^*, 1^*, 0^*\}, \{\Theta_{3^*}, 3^*, 1^*, 1^*\}, \{\Theta_{3^*}, 2^*, 1^*, 0^*\}, \{\Theta_{3^*}, 2^*, 1^*, 1^*\}, \\ & \left. \{\Theta_{3^*}, 1^*, 1^*, 0^*\}, \{\Theta_{3^*}, 1^*, 1^*, 1^*\} \right\} \end{aligned} \quad (54)$$

Omitting the part $j_{2R'-1}^*, \dots, j_3^*, j_1^*$ in (50), the set $\{\tau^*\}_0$ may be rewritten as $\{\tau^*\}_0 = \left\{ (\dots, 3^*, 3^*, 3^*), \{\Theta_{1^*}\}, \{\Theta_{3^*}\} \right\}$.

A $U_q(gl(2|2))$ -weight $\bar{h}^A = (\bar{h}_1^A, \bar{h}_2^A, \bar{h}_3^A, \bar{h}_4^A)$ for any configuration $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*) \in \Omega_A$ follows unambiguously from the $U_q(gl(2|2))$ -weight defined for a reference configuration. For the reference configuration $(\dots \otimes w_3 \otimes w_3^* \otimes w_3 \otimes w_3^*)$, two obvious choices \bar{h}^A and \bar{h}'^A are

$$\begin{aligned} \bar{h}_j^A &= 0 \quad \text{for } j = 1, 2, 3, 4 \\ \bar{h}_j'^A &= \delta_{j,4} - \delta_{j,3} \end{aligned} \quad (55)$$

Due to (19), (21) and (51), (54), the weights of the configurations with vanishing diagonal element of h_{CTM} evaluated with respect to \bar{h}^A are given by

$$(-t, t, t, -t), (-t, t+1, t, -t-1), (-t-1, t, t+1, -t-2), (-t-1, t+1, t+1, -t-3), \quad t = 0, 1, 2, \dots \quad (56)$$

The corner transfer matrix built from composite vertices of type B is analyzed in a completely analogous manner. A half-infinite configuration $(\dots \otimes w_{j_4}^* \otimes w_{j_3} \otimes w_{j_2}^* \otimes w_{j_1})$ may be denoted shortly by $(\dots, j_4^*, j_3, j_2^*, j_1)$. Again, configurations with entries $j_{2r}^* = 3^*$ and $j_{2r-1} = 3$ for almost all r are selected. The set of all these configurations is denoted by Ω_B . In the limit $q \rightarrow 0$, the diagonal elements $h_{(\dots, j_4^*, j_3, j_2^*, j_1); (\dots, j_4^*, j_3, j_2^*, j_1)}^{ren}$ of the corresponding corner transfer matrix Hamiltonian are given by

$$h_{(\dots, j_4^*, j_3, j_2^*, j_1); (\dots, j_4^*, j_3, j_2^*, j_1)}^{ren} = - \sum_{r=1}^{\infty} r (y_{j_{2r+1}, j_{2r-1}} + x_{j_{2r+2}, j_{2r}}) \quad (57)$$

with x_{j_1, j_2} and y_{j_1, j_2} specified by (49). The configuration $(\dots, 3^*, 3, 3^*, 3)$ is an eigenvector if the limit is performed for a lattice composed from elementary R-matrices $R_{WW}(z)$, $R_{W^*W^*}(z)$, $R_{WW^*}(q^{-2}w^{-1}z)$ and $R_{W^*W}(q^2wz)$. Clearly, the classification in terms of the sets $\{\tau^*\}_{-n}$ and $\{\tau\}_{-n}$ expressed in (51) and (54) simultaneously applies to the configurations $(\dots, j_5^*, j_6, j_3^*, j_4, j_1^*, j_2)$. Two reference weights \bar{h}^B and \bar{h}'^B for the configuration $(\dots \otimes w_3^* \otimes w_3 \otimes w_3^* \otimes w_3)$ are fixed by

$$\bar{h}_j^B = 0 \quad \text{for } j = 1, 2, 3, 4$$

$$\bar{h}_j'^B = \delta_{j,3} - \delta_{j,4} \quad (58)$$

With respect to \bar{h}'^B , the weights of the configurations in Ω_B with vanishing diagonal element of h_{CTM} are listed by

$$(-t, t, t+1, -t-1), (-t, t+1, t+1, -t-2), (-t-1, t, t+2, -t-3), (-t-1, t+1, t+2, -t-4) \quad (59)$$

with $t = 0, 1, 2, \dots$

For any module of $U_q(gl(2|2))$ or $U_q(\widehat{gl}(2|2))$, the eigenvalues of h_4 may be shifted by an arbitrary fixed value without modifying the action of the remaining generators. Not being of any relevance for the following, this freedom will not be mentioned further.

In context with quantum affine algebra models, the notation paths refers to eigenvectors of the corner transfer matrix Hamiltonian. For convenience, the configurations encountered in the present model may also be addressed to as paths. Similarly, x_{j_1, j_2} and y_{j_1, j_2} in (48), (49) may be called generalized energy functions.

IV. LEVEL-ONE MODULES

The four-tuples (56) can be viewed as the weights of the vectors contained in the one-dimensional $U_q(gl(2|2))$ -module $V_{\bar{\Lambda}_0}$ with highest weight $(0, 0, 0, 0)$ and the infinite-dimensional, irreducible $U_q(gl(2|2))$ -module $V_{\bar{\Lambda}_2 - \bar{\Lambda}_4}$ with highest weight $(0, 1, 0, -1)$. All vectors contained in $V_{\bar{\Lambda}_0}$ and $V_{\bar{\Lambda}_2 - \bar{\Lambda}_4}$ together with the corresponding eigenvalues of h_1, h_2, h_3, h_4 consistent with \bar{h}^A are listed by

$$\begin{array}{ll} \lambda_0 & (0, 0, 0, 0) \\ \lambda_2 & (0, 1, 0, -1) \\ f_2 \lambda_2 & (-1, 1, 1, -1) \\ (f_3 f_2)^m \lambda_2 & (-m, m+1, m, -m-1) \\ f_2 (f_3 f_2)^m \lambda_2 & (-m-1, m+1, m+1, -m-1) \\ f_1 f_2 \lambda_2 & (-1, 0, 1, -2) \\ f_1 (f_3 f_2)^m \lambda_2 & (-m, m, m, -m-2) \\ f_1 f_2 (f_3 f_2)^m \lambda_2 & (-m-1, m, m+1, -m-2) \end{array} \quad m = 1, 2, 3, \dots \quad (60)$$

Here λ_0 and λ_2 denote the highest weight vectors of $V_{\bar{\Lambda}_0}$ and $V_{\bar{\Lambda}_2 - \bar{\Lambda}_4}$ with the properties

$$e_j \lambda_0 = f_j \lambda_0 = 0 \quad j = 1, 2, 3 \quad (61)$$

and

$$e_j \lambda_2 = 0 \quad \text{for } j = 1, 2, 3 \quad f_j \lambda_2 = 0 \quad \text{for } j = 1, 3 \quad (62)$$

The weights collected in (60) may be attributed to the vectors of one reducible $U_q(gl(2|2))$ -module $\tilde{V}_{\bar{\Lambda}_0}$ with highest weight $(0, 0, 0, 0)$ rather than to two irreducible modules. $\tilde{V}_{\bar{\Lambda}_0}$ is characterized by the highest weight vector κ with the properties

$$h_j \kappa = 0 \quad \text{for } j = 1, 2, 3, 4 \quad e_j \kappa = 0 \quad \text{for } j = 1, 2, 3 \quad f_3 \kappa \neq 0 \quad f_1 \kappa = f_2 \kappa = 0 \quad (63)$$

Similarly, the level-one highest weight $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules $V(\Lambda_0)$ and $V(\Lambda_2 - \Lambda_4)$ may be regarded as the two irreducible submodules of a level-one module $\tilde{V}(\Lambda_0)$. The latter has a highest weight vector $\hat{\kappa}$ satisfying

$$h_j \hat{\kappa} = \delta_{j,0} \hat{\kappa} \quad \text{for } j = 0, 1, 2, 3, 4 \quad d \hat{\kappa} = 0 \quad (64)$$

and

$$e_j \hat{\kappa} = 0 \text{ for } j = 0, 1, 2, 3 \quad f_j \hat{\kappa} \neq 0 \text{ for } j = 0, 3 \quad f_1 \hat{\kappa} = f_2 \hat{\kappa} = 0 \quad (65)$$

Thus the action of f_j on $\hat{\kappa}$ provides all vectors in $\tilde{V}(\Lambda_0)$. The module $\tilde{V}(\Lambda_0)$ contains \tilde{V}_{Λ_0} as its zero-grade subspace. It turns out that the configuration space of the present model is more conveniently characterized in terms of a reducible module. For this reason, the following analysis refers to $\tilde{V}(\Lambda_0)$. In Sect. V, the decomposition into irreducible constituents is taken up again. The eigenvalues of h_j acting on a vector in a $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module may be denoted by \bar{h}_j , $j = 0, 1, 2, 3, 4$. At level one, they are related by $\bar{h}_0 = 1 - \bar{h}_1 - \bar{h}_2 - \bar{h}_3$. For brevity, in the following only the $U_q(gl(2|2))$ -weight $(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ of a vector in a level-one module will be indicated. The eigenvalue of the grading operator d on any weight vector of the module is referred to as its grade. According to (64), (65) and the defining relations (1), the module includes the two vectors

$$f_0 \hat{\kappa} \quad \text{and} \quad f_0 f_1 f_3 f_2 f_3 \hat{\kappa} \quad (66)$$

with grade -1 and weights $(1, 0, -1, 2)$ and $(0, 1, 0, -1)$, respectively. Restricted by the Serre relations (2), the action of f_1, f_2, f_3 on $f_0 \hat{\kappa}$ generates the vectors

$$\begin{array}{ll} (f_2 f_3)^m f_0 \hat{\kappa} & (-m+1, m, m-1, -m+2) \\ f_3 (f_2 f_3)^m f_0 \hat{\kappa} & (-m+1, m+1, m-1, -m+1) \\ f_1 (f_2 f_3)^m f_0 \hat{\kappa} & (-m+1, m-1, m-1, -m+1) \\ f_1 f_3 (f_2 f_3)^m f_0 \hat{\kappa} & (-m+1, m, m-1, -m) \\ f_2 (f_3 f_2)^m f_1 f_0 \hat{\kappa} & (-m, m-1, m, -m+1) \\ (f_3 f_2)^{m+1} f_1 f_0 \hat{\kappa} & (-m, m, m, -m) \\ f_1 f_2 (f_3 f_2)^{m+1} f_1 f_0 \hat{\kappa} & (-m-1, m-1, m+1, -m-1) \\ f_1 (f_3 f_2)^{m+2} f_1 f_0 \hat{\kappa} & (-m-1, m, m+1, -m-2) \\ (f_3 f_2)^{m+1} f_3 f_1 f_0 \hat{\kappa} & (-m, m+1, m, -m-1) \\ f_2 (f_3 f_2)^m f_3 f_1 f_0 \hat{\kappa} & (-m, m, m, -m) \\ f_1 (f_3 f_2)^{m+1} f_3 f_1 f_0 \hat{\kappa} & (-m, m, m, -m-2) \\ f_1 f_2 (f_3 f_2)^m f_3 f_1 f_0 \hat{\kappa} & (-m, m-1, m, -m-1) \end{array} \quad (67)$$

with $m = 0, 1, 2, 3, \dots$. The corresponding $U_q(gl(2|2))$ -weights are listed in the right column. Taking into account (51), (55) and (56), these weights are found in one-to-one correspondence with the weights of all configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with $(\dots, j_6, j_4, j_2) \in \{\tau\}_{-1}$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_0$. The level-one module $\tilde{V}(\Lambda_0)$ does not contain $f_0 f_3 \hat{\kappa}$ or $f_0 f_2 f_3 \hat{\kappa} = -f_2 f_0 f_3 \hat{\kappa}$. Including these would yield a module with more than two irreducible $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -submodules. Use of (11), (15) and (18) yields

$$(H_{-1}^1 + H_{-1}^3) \hat{\kappa} = q^{-\frac{1}{2}} (f_0 f_1 f_2 f_3 \hat{\kappa} + f_3 f_2 f_1 f_0 \hat{\kappa} - f_1 f_2 f_3 f_0 \hat{\kappa}) \quad (68)$$

Regarding $\tilde{V}(\Lambda_0)$ as a $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module, the rhs of (68) is set to zero. On $f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$, the action of f_1, f_2, f_3 produces some of the vectors in (67) and the further vectors

$$\begin{array}{ll} (f_2 f_3)^m f_0 f_1 f_3 f_2 f_3 \hat{\kappa} & (-m, m+1, m, -m-1) \\ f_3 (f_2 f_3)^m f_0 f_1 f_3 f_2 f_3 \hat{\kappa} & (-m, m+2, m, -m-2) \\ f_1 (f_2 f_3)^{m+1} f_0 f_1 f_3 f_2 f_3 \hat{\kappa} & (-m-1, m+1, m+1, -m-3) \end{array}$$

$f_1 f_3 (f_2 f_3)^{m+1} f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-1, m+2, m+1, -m-4)$
$(f_3 f_2)^{m+1} f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-1, m+2, m+1, -m-2)$
$f_2 (f_3 f_2)^m f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-1, m+1, m+1, -m-1)$
$f_1 (f_3 f_2)^{m+2} f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-2, m+2, m+2, -m-4)$
$f_1 f_2 (f_3 f_2)^{m+1} f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-2, m+1, m+2, -m-3)$
$(f_2 f_3)^m f_1 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-1, m, m+1, -m-2)$
$f_3 (f_2 f_3)^m f_1 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-1, m+1, m+1, -m-3)$
$f_1 (f_2 f_3)^{m+1} f_1 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-2, m, m+2, -m-4)$
$f_1 f_3 (f_2 f_3)^{m+1} f_1 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$(-m-2, m+1, m+2, -m-5)$

(69)

where $m = 0, 1, 2, 3, \dots$. Their weights given in the right column coincide with the weights of all configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with $(\dots, j_6, j_4, j_2) \in \{\tau\}_0$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_{-1}$. Due to the Serre relations (2), the action of f_0 does not give rise to other vectors than those collected in (67) and (69). Thus the weights of the configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with the diagonal element of h_{CTM} given by -1 and the weights of the vectors contained in the level-one module $\tilde{V}(\Lambda_0)$ at grade -1 are found in one-to-one correspondence. These weights may be listed separately for the irreducible submodules $V(\Lambda_0)$ and $V(\Lambda_2 - \Lambda_4)$. The irreducible submodule $V(\Lambda_0)$ contains fourteen weights at grade -1 . They are given by $(1, 0, -1, 2)$, $(1, \pm 1, -1, 1)$, $(1, 0, -1, 0)$, $(0, \pm 1, -1, 0)$, $(0, 0, 0, 0)$, $(0, \pm 1, 0, -1)$, $(-1, \pm 1, 1, -1)$, $(-1, 0, 1, 0)$ and $(-1, 0, 1, 2)$, where $(0, 0, 0, 0)$ occurs with multiplicity two. Excluding these fourteen weights from the set of all weights listed in the right columns of (67) and (69) yields the set of weights found for the $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module $V(\Lambda_2 - \Lambda_4)$ at grade -1 .

To specify the weights present in $\tilde{V}(\Lambda_0)$ at lower grades, it is useful to introduce sets $\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ of $U_q(gl(2|2))$ -weights:

$$\begin{aligned}
\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4) = & \left\{ (\bar{h}_1 - m, \bar{h}_2 + m, \bar{h}_3 + m, \bar{h}_4 - m), \right. \\
& (\bar{h}_1 - m, \bar{h}_2 + m + 1, \bar{h}_3 + m, \bar{h}_4 - m - 1), \\
& (\bar{h}_1 - m - 1, \bar{h}_2 + m, \bar{h}_3 + m + 1, \bar{h}_4 - m - 2), \\
& \left. (\bar{h}_1 - m - 1, \bar{h}_2 + m + 1, \bar{h}_3 + m + 1, \bar{h}_4 - m - 3) \right\}_{m \in \mathbb{N}_0}
\end{aligned}
\tag{70}$$

Then the weights at a given grade can be arranged in a finite number of sets $\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ with $\bar{h}_i \in \mathbb{Z}$. For example, the weights of the zero-grade subspace \tilde{V}_{Λ_0} yield the set $\sigma(0, 0, 0, 0)$. The weights listed in (67) and (69) form the sets $\sigma(1, 0, -1, 2)$, $\sigma(1, -1, -1, 1)$, $\sigma(0, -1, 0, 1)$ and $\sigma(0, 1, 0, -1)$, $\sigma(-1, 1, 1, -1)$, $\sigma(-1, 0, 1, -2)$, respectively. All weight vectors at grade $-n$ can be expressed as polynomials in f_1, f_2, f_3 acting on finitely many vectors $\alpha_t^{(-n)}$, $t = 1, 2, 3, \dots$, obtained by the action of f_0 on vectors of grade $-(n-1)$. The assignment of t is such that the closer the positions of the generators f_0 are to κ , the smaller the value of t is. Due to the Serre relations (2), in some cases polynomials acting on different $\alpha_{t_i}^{(-n)}$ with fixed n , $1 \leq i \leq i'$ and $t_1 < t_2 < \dots < t_{i'}$ are linear dependent. The $\alpha_{t_i}^{(-n)}$ and the polynomials can be chosen such that the polynomials on $\alpha_{t_i}^{(-n)}$ are linear independent for $1 \leq i \leq i' - 1$. Then only the $i' - 1$ weights corresponding to the latter are listed below. An example is provided by $f_1 f_0 f_1 f_3 f_2 f_3 f_2 f_1 f_0 \kappa$. Applying (2) repeatedly and taking into account the weight structure (67), (69) of the module at grade -1 yields $f_1 f_0 f_1 f_3 f_2 f_3 f_2 f_1 f_0 \kappa = f_1 f_2 f_1 (f_3 f_2 f_3 f_0 + [2] f_2 f_3 f_0 f_3 + f_2 f_3 f_0 f_3 - f_3 f_0 f_3 f_2) f_1 f_0 \kappa$. The second and third term can be rewritten using $[2] f_0 f_3 f_1 f_0 \kappa = -f_3 f_0 f_1 f_0 \kappa$ which follows from the first line of (2). Moreover, as easily seen from equation (68), the constraint $(H_{-1}^1 + H_{-1}^3) \kappa = 0$ implies $f_0 f_3 f_2 f_1 f_0 \kappa = f_0 f_1 f_2 f_3 f_0 \kappa$. Thus $f_1 f_0 f_1 f_3 f_2 f_3 f_2 f_1 f_0 \kappa = -f_1 f_2 f_1 f_3 f_0 f_1 f_2 f_3 f_0 \kappa$. The weight of this vector is then included in the weight sets associated to $f_0 f_1 f_2 f_3 f_0 \kappa$. The six vectors $\alpha_t^{(-2)}$ and the sets collecting the weights attributed to the vectors obtained from them by application of f_1, f_2, f_3 are given by the following table:

$\alpha_t^{(-2)} :$	sets $\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ attributed to $\alpha_i^{(-2)} :$
$\alpha_1^{(-2)} = f_0 f_1 f_0 \hat{\kappa}$	$\sigma(2, -1, -2, 3), \sigma(2, -2, -2, 2), \sigma(1, -1, -1, 3), \sigma(1, -2, -1, 2)$
$\alpha_2^{(-2)} = f_0 f_1 f_2 f_3 f_0 \hat{\kappa}$	$\sigma(1, 0, -1, 2), \sigma(1, -1, -1, 1), \sigma(0, -1, 0, 1)$
$\alpha_3^{(-2)} = f_0 f_1 f_3 f_2 f_3 f_0 \hat{\kappa}$	$\sigma(1, 1, -1, 1), \sigma(1, 0, -1, 0), \sigma(0, 1, 0, 1), \sigma(0, 0, 0, 0),$ $\sigma(0, 0, 0, 0), \sigma(0, -1, 0, -1), \sigma(-1, 0, 1, 0), \sigma(-1, -1, 1, -1)$
$\alpha_4^{(-2)} = f_0 f_1 f_3 f_2 f_3 f_2 f_1 f_0 \hat{\kappa}$	$\sigma(0, 0, 0, 0)$
$\alpha_5^{(-2)} = f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(0, 1, 0, -1), \sigma(-1, 1, 1, -1), \sigma(-1, 0, 1, -2)$
$\alpha_6^{(-2)} = f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(-1, 2, 1, -2), \sigma(-1, 1, 1, -3), \sigma(-2, 2, 2, -2), \sigma(-2, 1, 2, -3)$

(71)

Each weight vector listed in (71) and in (72)-(73) below occurs with multiplicity one. The weights attributed to $\alpha_{2t-1}^{(-2)}$ and $\alpha_{2t}^{(-2)}$ for $t = 1, 2, 3$ coincide with the weights of the configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with $(\dots, j_6, j_4, j_2) \in \{\tau\}_{t-3}$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_{1-t}$. At grade -3 , seventy-three sets $\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ and fifteen vectors $\alpha_t^{(-3)}$ are found:

$$\begin{array}{ll}
\alpha_t^{(-3)} : & \text{sets } \sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4) \text{ attributed to } \alpha_t^{(-3)} \\
\alpha_1^{(-3)} = f_0 f_1 f_0 f_1 f_0 \hat{\kappa} & \sigma(3, -2, -3, 4), \sigma(3, -3, -3, 3), \sigma(2, -2, -2, 4), \\
& \sigma(2, -3, -2, 3), \\
\alpha_2^{(-3)} = f_0 f_3 f_2 f_0 f_1 f_0 \hat{\kappa} & \sigma(2, 0, -2, 4), \sigma(2, -1, -2, 3), \sigma(1, -1, -1, 3), \\
& \sigma(1, -2, -1, 2), \sigma(0, -2, 0, 2) \\
\alpha_3^{(-3)} = f_0 f_1 f_2 f_3 f_0 f_1 f_0 \hat{\kappa} & \sigma(2, -1, -2, 3), \sigma(2, -2, -2, 2), \sigma(1, -1, -1, 3), \\
& \sigma(1, -2, -1, 2), \\
\alpha_4^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_0 \hat{\kappa} & \sigma(1, 0, -1, 2), \sigma(1, -1, -1, 1), \sigma(0, -1, 0, 1) \\
\alpha_5^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_0 f_1 f_0 \hat{\kappa} & \sigma(2, 0, -2, 2), \sigma(2, -1, -2, 1), \sigma(1, 0, -1, 2), \\
& \sigma(1, -1, -1, 1), \sigma(1, -1, -1, 1), \sigma(1, -2, -1, 0), \\
& \sigma(0, -1, 0, 1), \sigma(0, -2, 0, 0) \\
\alpha_6^{(-3)} = f_0 f_1 f_2 f_3 f_0 f_1 f_2 f_3 f_0 \hat{\kappa} & \sigma(1, 0, -1, 2), \sigma(1, -1, -1, 1), \sigma(0, 0, 0, 2), \\
& \sigma(0, -1, 0, 1) \\
\alpha_7^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_0 f_1 f_2 f_3 f_0 \hat{\kappa} & \sigma(1, 1, -1, 1), \sigma(1, 0, -1, 0), \sigma(0, 1, 0, 1), \\
& \sigma(0, 0, 0, 0), \sigma(0, 0, 0, 0), \sigma(0, -1, 0, -1), \\
& \sigma(-1, 0, 1, 0) \sigma(-1, -1, 1, -1) \\
\alpha_8^{(-3)} = f_0 f_1 f_3 f_2 f_1 f_2 f_0 f_1 f_3 f_2 f_3 f_0 \hat{\kappa} & \sigma(0, 0, 0, 0)
\end{array} \tag{72}$$

and

$\alpha_t^{(-3)} :$	sets $\sigma(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$ attributed to $\alpha_t^{(-3)}$
$\alpha_9^{(-3)} = f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_3 f_0 \hat{\kappa}$	$\sigma(1, 1, -1, 1), \sigma(1, 0, -1, 0), \sigma(0, 1, 0, 1),$ $\sigma(0, 0, 0, 0), \sigma(0, 0, 0, 0), \sigma(0, -1, 0, -1),$ $\sigma(-1, 0, 1, 0), \sigma(-1, -1, 1, -1)$
$\alpha_{10}^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_3 f_2 f_3 f_0 \hat{\kappa}$	$\sigma(0, 2, 0, 0), \sigma(0, 1, 0, -1), \sigma(-1, 2, 1, 0),$ $\sigma(-1, 1, 1, -1), \sigma(-1, 1, 1, -1) \sigma(1, 0, 1, -2),$ $\sigma(-2, 1, 2, -1), \sigma(-2, 0, 2, -2)$
$\alpha_{11}^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_2 f_1 f_0 f_1 f_3 f_2 f_3 f_0 \hat{\kappa}$	$\sigma(0, 1, 0, -1), \sigma(0, 0, 0, -2), \sigma(-1, 1, 1, -1),$ $\sigma(-1, 0, 1, -2)$
$\alpha_{12}^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(0, 2, 0, -2), \sigma(-1, 2, 1, -2), \sigma(-1, 1, 1, -3),$ $\sigma(-2, 1, 2, -3), \sigma(-2, 0, 2, -4)$
$\alpha_{13}^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(-1, 2, 1, -2), \sigma(-1, 1, 1, -3), \sigma(-2, 2, 2, -2),$ $\sigma(-2, 1, 2, -3)$
$\alpha_{14}^{(-3)} = f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(0, 1, 0, -1), \sigma(-1, 1, 1, -1), \sigma(-1, 0, 1, -2)$
$\alpha_{15}^{(-3)} = f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_3 f_2 f_3 f_2 f_0 f_1 f_3 f_2 f_3 \hat{\kappa}$	$\sigma(-2, 3, 2, -3), \sigma(-2, 2, 2, -4), \sigma(-3, 3, 3, -3),$ $\sigma(-3, 2, 3, -4)$

(73)

The weights attributed to the vectors $\alpha_{4t-3}^{(-3)}, \alpha_{4t-2}^{(-3)}, \alpha_{4t-1}^{(-3)}$ and $\alpha_{4t}^{(-3)}$ for $t = 1, 2$ coincide with the weights of the configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with $(\dots, j_6, j_4, j_2) \in \{\tau\}_{t-4}$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_{1-t}$. The weights related to $\alpha_9^{(-3)}, \alpha_{10}^{(-3)}, \alpha_{11}^{(-3)}$ are the weights of the configurations with $(\dots, j_6, j_4, j_2) \in \{\tau\}_{-1}$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_{-2}$. Finally, the weights of $\alpha_t^{(-3)}$ with $12 \leq t \leq 15$ correspond to the weights of the configurations with $(\dots, j_6, j_4, j_2) \in \{\tau\}_0$ and $(\dots, j_5^*, j_3^*, j_1^*) \in \{\tau^*\}_{-3}$. The tables (71), (72) and (73) result from imposing both the Serre relations (2) and the conditions $(H_{-n}^1 + H_{-n}^3) \hat{\kappa} = 0$ with $n = 1, 2$, for (71) and $n = 1, 2, 3$ for (72), (73).

In summary, for $n = 0, 1, 2, 3$, the weights of the vectors found in $\tilde{V}(\Lambda_0)$ at grade $-n$ and the weights of all configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with diagonal element of h_{CTM} given by $-n$ are in one-to-one correspondence. Provided that this remains true at any grade, the following statement holds.

Conjecture I: The character of $\tilde{V}(\Lambda_0)$ can be written

$$\begin{aligned}
ch_{\tilde{V}(\Lambda_0)}(\rho, p_0, p_1, p_2) &\equiv tr_{\tilde{V}(\Lambda_0)} \rho^d p_0^{\frac{1}{2}(h_1+h_2+h_3+h_4)} p_1^{\frac{1}{2}(h_1-h_2-h_3-h_4)} p_2^{-\frac{1}{2}(h_1+h_2-h_3-h_4)} = \\
&= \sum_{\{\dots, j_6, j_4, j_2\}} \rho^{-\sum_{r=1}^{\infty} r y_{j_{2r}+2, j_{2r}}} \prod_{j=0,1,2} p_j^{\sum_{r=1}^{\infty} \delta_{j_{2r}, j}} \cdot \sum_{\{\dots, j_5^*, j_3^*, j_1^*\}} \rho^{-\sum_{\bar{r}=1}^{\infty} \bar{r} x_{j_{2\bar{r}+1}, j_{2\bar{r}-1}}} \prod_{j'=0,1,2} p_{j'}^{-\sum_{\bar{r}=1}^{\infty} \delta_{j_{2\bar{r}-1}, j'}} \quad (74)
\end{aligned}$$

with $|p_1| > 1$.

As described in section III, the configurations in (74) are restricted by the requirement that $j_{2r} = 3 \forall r > R$ and $j_{2r+1}^* = 3^* \forall r > R \in \mathbb{N}$. The weights of the configurations are consistent with the reference weight \bar{h}^A of the configuration $(\dots, 3, 3^*, 3, 3^*)$ specified in (55).

Since each contribution to the sum (74) is positive, the conjecture states a character formula of quasiparticle type. Obviously, the modules $V(\Lambda_2 - \Lambda_4)$ and $\tilde{V}(\Lambda_0)$ do not satisfy the third requirement for weak integrability given in Sect. II. Consequently, the characters given in [35], [37] do not include the character of the $\widehat{gl}(2|2)$ -module corresponding to $V(\Lambda_2)$. In contrast, the irreducible module $V(\Lambda_0)$ is weakly integrable. The highest weight Λ_0 remains the same when changing the simple root system by odd reflections [35]. Thus the $U_q(\widehat{gl}(2|2))$ -module $V(\Lambda_0)$ is associated to the weakly integrable module F_0 in [35]. Unfortunately, in the configuration space the decomposition into irreducible components appears to be rather indirect. An expression for the character of $V(\Lambda_0)$ will be discussed in Sect. VII.

The R-matrix solution for the vector representation related to the standard system of simple roots has been given in [47]. Inspection of the appropriate $q \rightarrow 0$ -limits indicates that also in this case weight structures of nonintegrable modules are relevant to the description of the analogous inhomogeneous vertex model. Moreover, nonintegrable modules should be related to the analogous $U_q(\widehat{gl}(m|m'))$ -models with $m, m' > 0$.

Alternatively, the weights may be evaluated with respect to \bar{h}'^A . Then the weights of all configurations with vanishing diagonal element of h_{CTM} coincide with the weights of an infinite-dimensional, reducible $U_q(gl(2|2))$ -module $\tilde{V}_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$. Its highest weight vector ν is characterized by

$$\begin{aligned} h_j \nu &= \delta_{j,4} - \delta_{j,3} \\ e_j \nu &= 0 \quad \text{for } j = 1, 2, 3 \quad f_3 f_2 f_3 \nu \neq 0 \quad f_1 \nu = f_2 \nu = 0 \end{aligned} \quad (75)$$

$\tilde{V}_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$ may be decomposed into a four-dimensional irreducible $U_q(gl(2|2))$ -module $V_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$ with weights $(0, 0, -1, 1)$, $(0, 1, -1, 0)$, $(-1, 1, 0, 0)$ and $(-1, 0, 0, -1)$ and an infinite-dimensional irreducible module $V_{-\bar{\Lambda}_1 + 2\bar{\Lambda}_2 - \bar{\Lambda}_4}$ with weights $(-m, m+1, m-1, -m)$, $(-m-1, m+1, m, -m)$, $(-m, m, m-1, -m-1)$ and $(-m-1, m, m, -m-1)$, $m = 1, 2, 3, \dots$. $V_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$ and $V_{-\bar{\Lambda}_1 + 2\bar{\Lambda}_2 - \bar{\Lambda}_4}$ contain the highest weight vectors λ_{-3} and $\lambda_{-1,2^2}$ with $h_j \lambda_{-3} = (\delta_{j,4} - \delta_{j,3}) \lambda_{-3}$ and $h_j \lambda_{-1,2^2} = (2\delta_{j,2} - \delta_{j,1} - \delta_{j,4}) \lambda_{-1,2^2}$. The level-one $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ with its grade-zero subspace given by $\tilde{V}_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$ can be decomposed into the level-one irreducible modules $V(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ and $V(-\Lambda_1 + 2\Lambda_2 - \Lambda_4)$. The latter contain $V_{-\bar{\Lambda}_3 + \bar{\Lambda}_4}$ and $V_{-\bar{\Lambda}_1 + 2\bar{\Lambda}_2 - \bar{\Lambda}_4}$ as their grade-zero subspaces, respectively. An analysis based on the defining relations of $U_q(\widehat{sl}(2|2))$ reveals that shifting the weights of all vectors in $\tilde{V}(\Lambda_0)$ at grade $-n$ by $(0, 0, -1, 1)$ yields the complete set of weights found in $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ at grade $-n$, $n = 0, 1, 2, 3$. Thus, for the grades $-n \geq -3$, switching to the reference weight \bar{h}'^A amounts to replacing $\tilde{V}(\Lambda_0)$ by $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$. Above the eigenvalues of the grading operator are set to zero for the highest weight vectors of both modules $\tilde{V}(\Lambda_0)$ and $\tilde{V}(\Lambda_2 - \Lambda_4)$. A different choice results in a constant shift of all grades and it easily taken into account in each statement. The relation between the weight structures of $\tilde{V}(\Lambda_0)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ becomes apparent in the boson realization of both modules constructed in Sect. V.

From the results described above, an interpretation of the weights associated the half-infinite configurations $(\dots \otimes w_{j_3}^* \otimes w_{j_4} \otimes w_{j_1}^* \otimes w_{j_2})$ in Ω_B is readily obtained. As pointed out in the preceeding section, the classification of the configurations in Ω_A and Ω_B according to the diagonal elements of the corner transfer matrix Hamiltonians in the limit $q \rightarrow 0$ is achieved by the same sets $\{\tau\}_{-n}$ and $\{\tau^*\}_{-n}$. Choosing the reference weights \bar{h}^A and \bar{h}^B specified in (56) and (59), the weights of any two configurations $(\dots, j_3^*, j_4, j_1^*, j_2) \in \Omega_B$ and $(\dots, j_4, j_3^*, j_2, j_1^*) \in \Omega_A$ coincide. Hence, with \bar{h}^B the description by the weights of $\tilde{V}(\Lambda_0)$ holds for the configurations in Ω_B , too. The irreducible level-one $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules $V(\Lambda_1 + \Lambda_4)$ and $V(\Lambda_3 - \Lambda_4)$ relevant for the reference weight \bar{h}'^B have highest weight vectors $\hat{\lambda}_1$ and $\hat{\lambda}_3$ with the properties

$$\begin{aligned} h_j \hat{\lambda}_1 &= (\delta_{j,1} + \delta_{j,4}) \hat{\lambda}_1, \quad j = 0, 1, 2, 3, 4 \\ e_k \hat{\lambda}_1 &= 0 \quad \text{for } j = 0, 1, 2, 3 \quad f_k \hat{\lambda}_1 = 0 \quad \text{for } k = 0, 2, 3 \end{aligned} \quad (76)$$

and

$$\begin{aligned} h_j \hat{\lambda}_3 &= (\delta_{j,3} - \delta_{j,4}) \hat{\lambda}_3, \quad j = 0, 1, 2, 3, 4 \\ e_k \hat{\lambda}_3 &= 0 \quad \text{for } k = 0, 1, 2, 3 \quad f_k \hat{\lambda}_3 = 0 \quad \text{for } k = 0, 1, 2 \end{aligned} \quad (77)$$

Applying a uniform shift by $(0, 0, -1, 1)$, the weights of all vectors present in the grade-zero subspace of $V(\Lambda_3 - \Lambda_4)$ coincide with the weights collected in (56). An automorphism $\bar{\varsigma}$ of $U'_q(\widehat{gl}(2|2))$ is given by

$$\begin{aligned}
\bar{\varsigma}(e_0) &= e_3 & \bar{\varsigma}(e_1) &= e_2 & \bar{\varsigma}(e_2) &= e_1 & \bar{\varsigma}(e_3) &= e_0 \\
\bar{\varsigma}(f_0) &= f_3 & \bar{\varsigma}(f_1) &= f_2 & \bar{\varsigma}(f_2) &= f_1 & \bar{\varsigma}(f_3) &= f_0 \\
\bar{\varsigma}(h_0) &= h_3 & \bar{\varsigma}(h_1) &= h_2 & \bar{\varsigma}(h_2) &= h_1 & \bar{\varsigma}(h_3) &= h_0
\end{aligned} \tag{78}$$

$\bar{\varsigma}$ gives rise to the isomorphism ς of the vector spaces $V(\Lambda_0)$ ($V(\Lambda_2 - \Lambda_4)$) and $V(\Lambda_3 - \Lambda_4)$ ($V(\Lambda_1 + \Lambda_4)$) via

$$\begin{aligned}
\varsigma(\hat{\lambda}_0) &= \hat{\lambda}_3 & \varsigma(au) &= \bar{\varsigma}(a)\varsigma(u) \quad \text{with } u \in V(\Lambda_0) \\
\varsigma(\hat{\lambda}_2) &= \hat{\lambda}_1 & \varsigma(au) &= \bar{\varsigma}(a)\varsigma(u) \quad \text{with } u \in V(\Lambda_2 - \Lambda_4)
\end{aligned} \tag{79}$$

with $a \in U'_q(\widehat{gl}(2|2))$. Here $\hat{\lambda}_0$ and $\hat{\lambda}_2$ denote the highest weight vectors of $V(\Lambda_0)$ and $V(\Lambda_2 - \Lambda_4)$. Reducible modules may be composed from $V(\Lambda_1 + \Lambda_4)$ and $V(\Lambda_3 - \Lambda_4)$ in various ways. Obviously, the isomorphism ς may be applied to the module $\tilde{V}(\Lambda_0)$. Another reducible module $\tilde{V}(\Lambda_1 + \Lambda_4)$ is characterized by the vector $\hat{\mu}$ with the properties

$$\begin{aligned}
h_j \hat{\mu} &= (\delta_{j,1} + \delta_{j,4}) \hat{\mu}, \quad j = 0, 1, 2, 3, 4 \\
e_j \hat{\mu} &= 0 \quad \text{for } j = 1, 2, 3 & f_0 \hat{\mu} = f_2 \hat{\mu} = 0 & E_1^{3,-} f_2 f_1 \hat{\mu} = E_1^{2,-} f_3 f_1 \hat{\mu} = q e_0 \hat{\mu} \neq 0
\end{aligned} \tag{80}$$

The highest weight properties are imposed on the vector $e_0 \hat{\mu}$:

$$e_j e_0 \hat{\mu} = 0 \quad j = 0, 1, 2, 3 \tag{81}$$

With this choice, all three reducible modules $\tilde{V}(\Lambda_0)$, $\tilde{V}(\Lambda_1 + \Lambda_4)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ have simple realizations in terms of the same bosonization scheme given in Sect. V. A second automorphism $\bar{\varrho}$ of $U'_q(\widehat{gl}(2|2))$ is defined by

$$\begin{aligned}
\bar{\varrho}(e_0) &= e_2 & \bar{\varrho}(e_1) &= e_3 & \bar{\varrho}(e_2) &= e_0 & \bar{\varrho}(e_3) &= e_1 \\
\bar{\varrho}(f_0) &= f_2 & \bar{\varrho}(f_1) &= f_3 & \bar{\varrho}(f_2) &= f_0 & \bar{\varrho}(f_3) &= f_1 \\
\bar{\varrho}(h_0) &= h_2 & \bar{\varrho}(h_1) &= h_3 & \bar{\varrho}(h_2) &= h_0 & \bar{\varrho}(h_3) &= h_1
\end{aligned} \tag{82}$$

$\bar{\varrho}$ yields an automorphism ϱ of $\tilde{V}(\Lambda_0)$ via

$$\varrho(\hat{\mu}) = f_3 \hat{\mu} \quad \varrho(au) = \bar{\varrho}(a) \varrho(u) \quad \text{with } a \in U'_q(\widehat{sl}(2|2)), u \in \tilde{V}(\Lambda_0) \tag{83}$$

Isomorphisms of the irreducible modules $V(\Lambda_0)$, $V(\Lambda_1 + \Lambda_4)$ or $V(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ and $V(\Lambda_2 - \Lambda_4)$, $V(\Lambda_3 - \Lambda_4)$ or $V(-\Lambda_1 + 2\Lambda_2 - \Lambda_4)$ are obtained setting

$$\varrho(\hat{\lambda}_0) = \hat{\lambda}_2 \quad \varrho(\hat{\lambda}_1) = \hat{\lambda}_3 \quad \varrho(\hat{\lambda}_{-3}) = \hat{\lambda}_{-1,2^2} \tag{84}$$

where $\hat{\lambda}_{-3}$ and $\hat{\lambda}_{-1,2^2}$ denote the highest weight vectors of $V(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ and $V(-\Lambda_1 + 2\Lambda_2 - \Lambda_4)$, respectively. Relations between the weight structures of $\tilde{V}(\Lambda_0)$ and $\tilde{V}(\Lambda_1 + \Lambda_4)$ follow from the isomorphisms ς and ϱ providing maps between the irreducible components of these modules. They can also be inferred from the boson realizations of the reducible modules.

Assuming that the above correspondences hold at any grade, the set of all infinite configurations $\dots \otimes w_{j_3} \otimes w_{j_2}^* \otimes w_{j_1} \otimes w_{j_0}^* \otimes w_{j_{-1}} \otimes w_{j_{-2}}^* \otimes \dots$ with $j_r = 3$ for almost all r may be interpreted as a level-zero module. Depending on the choice of the reference weight, it is given by

$$\tilde{V}(\Lambda_0) \otimes \tilde{V}(\Lambda_1 + \Lambda_4)^{*S} \quad \text{or} \quad \tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4) \otimes \tilde{V}(\Lambda_0)^{*S} \tag{85}$$

Analogously, the set of all configurations $\dots \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^* \otimes w_{j_0} \otimes w_{j_{-1}}^* \otimes w_{j_{-2}} \otimes \dots$ with $j_r = 3$ for almost all r can be viewed as

$$\tilde{V}(\Lambda_0) \otimes \tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)^{*S} \quad \text{or} \quad \tilde{V}(\Lambda_1 + \Lambda_4) \otimes \tilde{V}(\Lambda_0)^{*S} \tag{86}$$

In the following sections, explicit notation of the weight Λ_4 will be omitted for brevity.

V. BOSONIZATION

This section provides a boson realization of $U_q(\widehat{gl}(2|2))$ at level one similar to the construction presented in [43] for $U_q(\widehat{sl}(M+1|N+1))$ with the standard choice of simple roots. The $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules $\tilde{V}(\iota_{II'}\Lambda_{I'})$ with $I, I' = 0, 1, 3$ and $\iota_{0I} = \delta_{I,0}$, $\iota_{1,I} = \delta_{I,1}$, $\iota_{3,I} = 2\delta_{I,0} - \delta_{I,3}$ are expressed as suitable restrictions of a Fock space.

Bosonization of the currents $\Psi^{j,\pm}(z)$ at level one requires four sets $\{\varphi^j, \varphi_0^j, \varphi_n^j, j = 1, 2, 3, 4; n \in \mathbb{Z}\}$ of deformed bosonic oscillators with commutation relations

$$\begin{aligned} [\varphi_n^j, \varphi_m^k] &= \delta_{j,k} \delta_{n+m,0} \frac{[n]^2}{n} \quad n, m \neq 0 \\ [\varphi_n^j, \varphi_0^k] &= i\delta_{j,k} \end{aligned} \quad (87)$$

In terms of the oscillators (87), the currents $\Psi^{j,\pm}(z)$ are realized by

$$\begin{aligned} \Psi^{4,+}(z) &= q^{\varphi_0^1 - i\varphi_0^4} \exp\left((q - q^{-1}) \sum_{n>0} (\varphi_n^1 - i\varphi_n^4) z^{-n}\right) \\ \Psi^{4,-}(z) &= q^{-\varphi_0^1 + i\varphi_0^4} \exp\left(-(q - q^{-1}) \sum_{n>0} (\varphi_{-n}^1 - i\varphi_{-n}^4) z^n\right) \end{aligned} \quad (88)$$

and

$$\begin{aligned} \Psi^{j,+}(z) &= q^{-i^j(\varphi_0^{j+1} + i\varphi_0^j)} \exp\left(-(q - q^{-1}) \sum_{n>0} i^j(\varphi_n^{j+1} + i\varphi_n^j) z^{-n}\right) \\ \Psi^{j,-}(z) &= q^{i^j(\varphi_0^{j+1} + i\varphi_0^j)} \exp\left((q - q^{-1}) \sum_{n>0} i^j(\varphi_{-n}^{j+1} + i\varphi_{-n}^j) z^n\right) \end{aligned} \quad (89)$$

for $j = 1, 2, 3$. Equations (88) and (89) imply

$$h_4 = \varphi_0^1 - i\varphi_0^4, \quad h_j = -i^j(\varphi_0^{j+1} + i\varphi_0^j) \quad \text{for } j = 1, 2, 3 \quad (90)$$

Combining each set of oscillators in the deformed free field

$$\varphi_n^{j,\pm}(z) = \varphi_n^j - i\varphi_0^j \ln z + i \sum_{n \neq 0} \frac{q^{\mp \frac{1}{2}|n|}}{[n]} \varphi_n^j z^{-n} \quad j = 1, 2, 3, 4 \quad (91)$$

the dependence of the bosonized currents $E^{k,\pm}(z)$ on the sets (87) can be expressed by

$$\begin{aligned} E^{j,+}(z) &= : \exp\left(-i^{j+1}(\varphi^{j+1,+}(z) + i\varphi^{j,+}(z))\right) : \exp(i\pi\delta_{j,1}\varphi_0^3) X^{j,+}(z) \\ E^{j,-}(z) &= : \exp\left(i^{j+1}(\varphi^{j+1,-}(z) + i\varphi^{j,-}(z))\right) : \exp(-i\pi\delta_{j,1}\varphi_0^3) X^{j,-}(z) \end{aligned} \quad (92)$$

for $j = 1, 2, 3$. The additional parts $X^{j,\pm}(z)$ are needed to impose relations (12)-(13). Their construction involves two further fields

$$\beta^r(z) = \beta^r - i\beta_0^r \ln z + i \sum_{n \neq 0} \frac{1}{n} \beta_n^r z^{-n} \quad (93)$$

with

$$\begin{aligned} [\beta_n^r, \beta_m^s] &= -n\delta_{r,s}\delta_{n+m,0} \\ [\beta^r, \beta_0^s] &= -i\delta_{r,s} \quad r, s = 1, 2 \end{aligned} \quad (94)$$

A suitable realization of $X^{j,\pm}(z)$ in terms of these fields is provided by

$$\begin{aligned}
-X^{1,-}(z) &= X^{2,+}(z) = \frac{1}{z(q-q^{-1})} \left(: \exp(\beta^1(q^{-1}z)) : - : \exp(\beta^1(qz)) : \right) \\
X^{1,+}(z) &= X^{2,-}(z) = : \exp(-\beta^1(z)) : \\
X^{3,+}(z) &= \frac{1}{z(q-q^{-1})} \left(: \exp(\beta^2(q^{-1}z)) : - : \exp(\beta^2(qz)) : \right) \\
X^{3,-}(z) &= : \exp(-\beta^2(z)) :
\end{aligned} \tag{95}$$

Finally, the expression for the grading operator d is inferred from the defining relations (14) and (89)-(95):

$$d = -\frac{1}{2} \sum_{j=1}^4 (\varphi_0^j)^2 - \sum_{j=1}^4 \sum_{n>0} \frac{n^2}{[n]^2} \varphi_{-n}^j \varphi_n^j + \frac{1}{2} \sum_{r=1,2} \beta_0^r (\beta_0^r - i) + \sum_{r=1,2} \sum_{n>0} \beta_{-n}^r \beta_n^r \tag{96}$$

The highest weight vectors $\hat{\kappa}$ and $\hat{\nu}$ of the reducible modules $\tilde{V}(\Lambda_0)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3)$ may be realized by

$$\hat{\kappa} = e^{\beta^2} |0\rangle \quad \hat{\nu} = e^{-\varphi^4 + 2\beta^2} |0\rangle \tag{97}$$

Furthermore, the vector $\hat{\mu} \in \tilde{V}(\Lambda_1)$ with properties (80) and (81) can be expressed by

$$\hat{\mu} = e^{i\varphi^1 + \beta^2} |0\rangle \tag{98}$$

Here the boson Fock vacuum $|0\rangle$ is characterized by

$$\varphi_0^j |0\rangle = \beta_0^r |0\rangle = 0 \quad \varphi_n^j |0\rangle = \beta_n^r |0\rangle = 0 \quad \forall n > 0, j = 1, 2, 3, 4, r = 1, 2 \tag{99}$$

The vector $e^{\beta^2} |0\rangle$ satisfies the highest weight properties $E_n^{k,+} e^{\beta^2} |0\rangle = 0 \forall n \geq 0$ and $E_n^{k,-} e^{\beta^2} |0\rangle = 0 \forall n > 0, k = 1, 2, 3$. From (89)-(95), all descendants of grade zero are listed by

$$\begin{aligned}
(E_0^{2,-} E_0^{3,-})^n \hat{\kappa} &\sim e^{n(\varphi^2 + \varphi^4 - \beta^1 - \beta^2) + \beta^2} |0\rangle \\
E_0^{3,-} (E_0^{2,-} E_0^{3,-})^n \hat{\kappa} &\sim e^{n(\varphi^2 + \varphi^4 - \beta^1 - \beta^2) + i\varphi^3 + \varphi^4} |0\rangle \\
E_0^{1,-} (E_0^{2,-} E_0^{3,-})^{n+1} \hat{\kappa} &\sim e^{n(\varphi^2 + \varphi^4 - \beta^1 - \beta^2) - i\varphi^1 + \varphi^4} |0\rangle \\
E_0^{1,-} E_0^{3,-} (E_0^{2,-} E_0^{3,-})^{n+1} \hat{\kappa} &\sim e^{n(\varphi^2 + \varphi^4 - \beta^1 - \beta^2) - i\varphi^1 + i\varphi^3 + 2\varphi^4} |0\rangle
\end{aligned} \tag{100}$$

with $n = 0, 1, 2, \dots$. The vectors (100) coincide with the grade-zero subspace of $\tilde{V}(\Lambda_0)$. Thus a suitable subspace of the Fock space \mathcal{F}_0 related to $e^{\beta^2} |0\rangle$ can be expected to furnish a boson realization of $\tilde{V}(\Lambda_0)$. Similar considerations apply to $\tilde{V}(\Lambda_1)$ and $\tilde{V}(2\Lambda_0 - \Lambda_3)$. According to the boson realizations of the currents (89)-(92), the Fock spaces \mathcal{F}_I are

$$\mathcal{F}_I = \mathbb{C}[\tilde{\varphi}_{-1}^r, \beta_{-1}^s, \tilde{\varphi}_{-2}^{r'}, \beta_{-2}^{s'}, \dots] \otimes \left(\bigoplus_{s_1, s_2, s_3 \in \mathbb{Z}} \mathbb{C} e^{s_1(i\varphi^1 + \varphi^2 - \beta^1) + s_2(\varphi^2 - i\varphi^3 - \beta^1) + s_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^2 + \alpha_I} |0\rangle \right) \tag{101}$$

with

$$\alpha_0 = 0 \quad \alpha_1 = \varphi^4 - \beta^2 \quad \alpha_3 = -\varphi^4 + \beta^2 \tag{102}$$

and

$$\begin{aligned}
\tilde{\varphi}_{-n}^1 &\equiv \varphi_{-n}^1 - i\varphi_{-n}^2 + \varphi_{-n}^3 - i\varphi_{-n}^4 \\
\tilde{\varphi}_{-n}^2 &\equiv i\varphi_{-n}^2 + \varphi_{-n}^3
\end{aligned} \tag{103}$$

Only two linear combinations $\tilde{\varphi}_{-n}^1$ and $\tilde{\varphi}_{-n}^2$ are required since the Fock space \mathcal{F}_I should contain a realization of a $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module rather than a $U_q(\widehat{gl}(2|2))$ -module.

The construction (92) implies the existence of two screening operators $\eta^r(z) = \sum_{n \in \mathbb{Z}} \eta_n^r z^{-n-1} =: e^{-\beta^r(z)}$: for $r = 1, 2$.

The particular form of the highest weight vectors (97) and of (98) as well as the fact that η_0^1 commutes with all generators of $U_q(\widehat{gl}(2|2))$ indicate that the boson realizations of $\tilde{V}(\iota_{II'}\Lambda_{I'})$ are contained in the restricted Fock spaces

$$Ker_{\eta_0^1} \mathcal{F}_I \quad I = 0, 1, 3 \quad (104)$$

With the action of the grading operator d on the vacuum $|0\rangle$ fixed by $d|0\rangle = 0$, its realization by (96) leads to

$$\begin{aligned} d \left(e^{s_1(i\varphi^1 + \varphi^2 - \beta^1) + s_2(\varphi^2 - i\varphi^3 - \beta^1) + s_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^2 + \alpha_I} |0\rangle \right) = \\ = -\frac{1}{2} \left(s_1(s_1 + 1) + (s_2 - s_3)(s_2 - s_3 + 1) - \delta_{I,1} + \delta_{I,3} \right) \cdot \\ \cdot e^{s_1(i\varphi^1 + \varphi^2 - \beta^1) + s_2(\varphi^2 - i\varphi^3 - \beta^1) + s_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^2 + \alpha_I} |0\rangle, \quad I = 0, 1, 3 \end{aligned} \quad (105)$$

Equations (90) and (105) allow to collect the weights of all vectors in $Ker_{\eta_0^1} \mathcal{F}_I$ at a given grade. At grades $-1, -2, -3$, the author finds a one-to-one correspondence of the weights in $Ker_{\eta_0^1} \mathcal{F}_0$ with the weights of $\tilde{V}(\Lambda_0)$ listed in the previous section ((60), (67), (69), (71)-(73)). This results supports the conjecture that (104) provides the boson realization of the level-one modules $\tilde{V}(\iota_{II'}\Lambda_{I'})$:

$$\tilde{V}(\iota_{II'}\Lambda_{I'}) = Ker_{\eta_0^1} \mathcal{F}_I \quad I = 0, 1, 3 \quad (106)$$

Introduction of a further fermionic field $\xi^2(z) = \sum_{n \in \mathbb{Z}} \xi_n^2 z^{-n} =: e^{\beta^2(z)}$: allows for a convenient decomposition of the modules $\tilde{V}(\iota_{II'}\Lambda_{I'})$ into irreducible components. Relations (94) lead to $\{\xi_n^r, \eta_m^s\} = \delta_{r,s} \delta_{n+m,0}$ and $\{\xi_n^r, \xi_m^s\} = \{\eta_n^r, \eta_m^s\} = 0$. In terms of ξ_0^2 and η_0^2 , direct sum decompositions are provided by $Ker_{\eta_0^1} \mathcal{F}_I = \xi_0^2 \eta_0^2 Ker_{\eta_0^1} \mathcal{F}_I \oplus \eta_0^2 \xi_0^2 Ker_{\eta_0^1} \mathcal{F}_I$. The second part constitutes boson realizations of the irreducible $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules $V(\Lambda_2)$, $V(\Lambda_3)$ and $V(-\Lambda_1 + 2\Lambda_2)$ specified in the previous section:

$$V(\Lambda_2) = Ker_{\eta_0^2} Ker_{\eta_0^1} \mathcal{F}_0 \quad V(\Lambda_3) = Ker_{\eta_0^2} Ker_{\eta_0^1} \mathcal{F}_1 \quad V(-\Lambda_1 + 2\Lambda_2) = Ker_{\eta_0^2} Ker_{\eta_0^1} \mathcal{F}_3 \quad (107)$$

Expressions for the irreducible $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules $V(\iota_{II'}\Lambda_{I'})$ are obtained as

$$V(\iota_{II'}\Lambda_{I'}) = \eta_0^2 Ker_{\eta_0^1} \mathcal{F}_I \quad I = 0, 1, 3 \quad (108)$$

Boson realizations provide an efficient tool to obtain character formulae. In [45], character expressions for the modules $V(\Lambda_I)$, $V(\Lambda_2)$ and $V(-\Lambda_1 + 2\Lambda_2)$ are found. The character expressions are sums of both positive and negative contributions. Hence, unlike the character expressions in [37] and formula (3.14) in [35], there are not of quasiparticle type. According to (108), the highest weight vector of $V(2\Lambda_0 - \Lambda_3)$ is realized by $\beta_{-1}^2 e^{-\varphi^4 + \beta^2} |0\rangle$. The module $V(2\Lambda_0 - \Lambda_3)$ is not considered in [45], since there analysis is restricted to highest weight vector given by pure exponentials in φ^j and β^1 acting on the Fock vacuum.

VI. BORDER STRIPES AND LEVEL-ZERO MODULES OF $U'_q(\widehat{SL}(2|2))$

In Sect. IV, the space of half-infinite configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ has been related to one reducible or two irreducible modules of $U_q(\widehat{sl}(2|2))/\mathcal{H}$ with level one. The aim of section VIII is a description of the half-infinite configurations in terms of infinitely many level-zero modules of $U_q(\widehat{sl}(2|2))$. For this purpose, classification by border stripes offers a useful tool.

Following [51], a skew Young diagram $\lambda \setminus \mu$ may be associated to any two partitions λ, μ with $\lambda_i \geq \mu_i \forall i$. It consists of all squares with edges $(i-1, j-1)$, $(i-1, j)$, (i, j) and $(i, j-1)$ with the pair i, j satisfying the property $\lambda_j \geq i > \mu_j$. A connected skew diagram without 2×2 blocks of boxes is called a border strip. Below, two sets of numbers $\{n_i\}_{1 \leq i \leq R}$ and $\{m_i\}_{1 \leq i \leq R}$ with $R \geq 1$, $n_i > 0$, $m_i > 0$ for $i < R$ and $m_R \geq 0$ will be used to characterize a given border strip. If $R > 1$, the parameters $\{n_i; m_i\}$ and the partitions $(\lambda_1, \lambda_2, \dots, \lambda_{K+1})$ and $(\mu_1, \mu_2, \dots, \mu_{K+1})$ with $K = \sum_{i=1}^{R-1} n_i$ are related by

$$\lambda_1 = \sum_{i=1}^R m_i$$

$$\lambda_2 = 1 + \sum_{i=2}^R m_i \quad \text{for } 1 < j \leq 1 + n_1$$

$$\lambda_j = 1 + \sum_{i=S}^R m_i \quad \text{for } 1 + \sum_{i=1}^{S-2} n_i < j \leq 1 + \sum_{i=1}^{S-1} n_i, \quad 3 \leq S \leq R$$

$$\mu_{K+1} = 0$$

$$\mu_j = \sum_{i=2}^R m_i \quad \text{for } 0 < j \leq n_1$$

$$\mu_j = \sum_{i=S}^R m_i \quad \text{for } \sum_{i=1}^{S-2} n_i < j \leq \sum_{i=1}^{S-1} n_i, \quad 3 \leq S \leq R \quad (109)$$

In the case $R = 1$, this reduces to $\lambda_1 = M$ and $\mu_1 = 0$. Thus $n_i + 1$ denotes the number of boxes in the i -th column containing more than one box, counted from the right. Similarly, $m_i + 1$ is the number of boxes in the i -th row with more than two boxes, counted from the top to the bottom of the border strip. $m_R + 1$ and m_1 count the number of boxes in the lowest and in the uppermost row, respectively. The total number of boxes is given by $M = m_R + \sum_{i=1}^{R-1} (n_i + m_i)$. Fig. 3 illustrates an example.

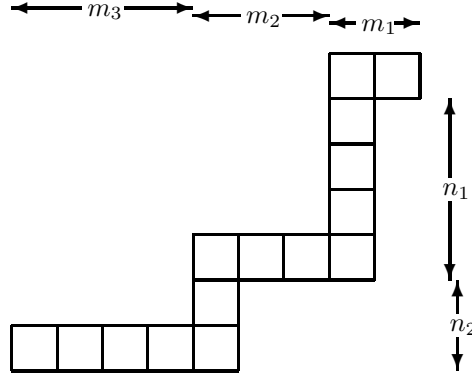


Fig. 3: Border strip with $R = 3$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$ and $n_1 = 4$, $n_2 = 2$

In contrast to the case of quantum affine algebras, the length of a column is not restricted. From a particular border strip, a semi-standard super tableau for $U'_q(\widehat{sl}(2|2))$ is obtained by attributing a number 0, 1, 2 or 3 to each of its boxes according to three rules. Within each row (column), the numbers are weakly decreasing from the left to the right (from the top to the bottom). Second, any row receives each of the numbers 1 and 3 at most once. Third, any column obtains each of the numbers 0 and 2 at most once. These three rules allow for two labelings involving only the numbers 0 and 1. Each column with $n_i + 1 > 1$ boxes obtains the value 1 in the upper n_i boxes. Attributing the number 0 to all remaining boxes yields a labeling referred to as top labeling below. A second labeling without the numbers 2 and 3 follows from the top labeling by replacing the number 0 in the lowest leftmost box of the border strip by 1. A semi-standard super tableau according to the three rules specified above will be called a type A labeling in the remainder. To the number l in a type A labeling, the $U_q(gl(2|2))$ -weight of w_l given in (19), (22) may be attributed. A $U_q(gl(2|2))$ -weight for a given labeling is then introduced as the sum over all weights related to its

numbers. Denoting the multiplicity of a number l in a given labeling T_A of type A by $N_l(T_A)$, the (supersymmetric) skew Schur function $s_{\lambda \setminus \mu}(\mathbf{p})$ is defined by

$$s_{\lambda \setminus \mu}(\mathbf{p}) \equiv \sum_{T_A \in T_A(\lambda \setminus \mu)} p_0^{N_0(T_A)} p_1^{N_1(T_A)} p_2^{N_2(T_A)} p_3^{N_3(T_A)} \quad (110)$$

where $T_A(\lambda \setminus \mu)$ denotes the set of all type A labelings of the border strip characterized by the partitions λ, μ given in (109). The skew Schur function can be expressed as

$$s_{\lambda \setminus \mu}(\mathbf{p}) = \det(e_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq K+1} \quad (111)$$

where $e_0 = 1$, $e_{-m} = 0$ and

$$e_m = p_0^{m-1}(p_0 + p_1) + \frac{p_2 + p_3}{p_2 - p_0}(p_2^m - p_0^m + p_1 p_2^{m-1} - p_1 p_0^{m-1}) \quad (112)$$

for $m = 1, 2, 3, \dots$. Equation (111) corresponds to the expression proved for the skew Schur functions attributed to $U_q(\widehat{sl}(n))$ in [51].

For a given border stripe, a subset of its semi-standard super tableaux is obtained by discarding any labeling of the boxes with 3 in the lowest leftmost box. In the following, this subset will be referred to as the restricted (type A) labelings. Expressions (48) and (49) for the diagonal elements of the CTM Hamiltonian reveal a one-to-one correspondence between the restricted labelings and the sets of configurations $\{\tau\}_n$ with $n < 0$. Formally, the unique configuration in $\{\tau\}_0$ may be related to the value $R = 0$. The boxes in a border strip of total length M are counted starting from its lower left end. Then, for a given labeling subject to the above rules, the number in the k -th box is associated with the entry $j_{2(M-k+1)}$ of a component (\dots, j_6, j_4, j_2) in $\{\tau\}_{n_{\{n_i; m_i\}}}$. The remaining entries are fixed by $j_{2r} = 3 \ \forall r > M$. A restricted labeling yields $j_{2M} \neq 3$. The number $n_{\{n_i; m_i\}}$ denoting the contribution of this particular component to the diagonal element of the CTM-Hamiltonian is given by

$$n_{\{n_i; m_i\}} = -M - \sum_{i=1} \left\{ \frac{1}{2} m_i(m_i - 1) + m_{i+1} \sum_{j=1}^i (m_j + n_j) \right\} \quad (113)$$

If $m_R > 0$, the weights of the discarded labelings can be written

$$(M - 1 - t_2, -K + t_3 - t_1, 1 + t_2, M - K - 2 - t_1 - t_3) \quad (114)$$

with suitable values of $t_l > 0$. In the case $m_R = 0$, the dropped weights have the form

$$(M - n_{R-1} - 1 - t_2, -K + n_{R-1} + t_3 - t_1, n_{R-1} + 1 + t_2, M - K - n_{R-1} - 2 - t_1 - t_3) \quad (115)$$

A restricted skew Schur function $s'_{\lambda \setminus \mu}(\mathbf{p})$ may be defined by replacing the sum in (110) by the sum over all restricted type A labelings of the border strip. Then the determinant formula reads

$$s'_{\lambda \setminus \mu}(\mathbf{p}) = \det(E')_{1 \leq i, j \leq K+1} \quad (116)$$

where $E'_{i,j} = e_{\lambda_i - \mu_j - i + j}$ for $j \leq K$ and $E'_{i,K+1} = \tilde{e}_{\lambda_i - i + K+1}$ with

$$\tilde{e}_m = p_0^{m-1}(p_0 + p_1) + \frac{p_2}{p_2 - p_0}(p_2^m - p_0^m + p_1 p_2^{m-1} - p_1 p_0^{m-1}) \quad (117)$$

Including the labelings with the number 3 in the lowest leftmost box, the semi-standard tableaux can be related to level-zero modules of $U'_q(\widehat{sl}(2|2))$. To this aim, a complex number x_k is attributed to the k -th box of the border strip (see Fig. 4). With the abbreviations

$$\begin{aligned} t_0 &= m_R + n_{R-1} \\ t_r &= m_R + n_{R-r-1} + \sum_{i=1}^r (n_{R-i} + m_{R-i}) \quad 1 \leq r \leq R-2 \end{aligned} \quad (118)$$

their relations are expressed by

$$x_{t_r+1} = q^{-2(m_{R-r-1}+1-\delta_{R-r,2})} x_{t_r} = q^{-2(t_r-m_R+m_{R-r-1}-\delta_{R-r,2})} x_1 \quad \text{for } 0 \leq r \leq R-2 \quad (119)$$

$$\begin{aligned} x_{t_r+1+k} &= q^{2k} x_{t_r+1} \quad \text{for } 1 \leq k \leq t_{R-r-1} - \delta_{R-r,2}, \quad 0 \leq r \leq R-2 \\ x_{k+1} &= q^{2k} x_k \quad \text{for } 1 \leq k \leq m_R - \delta_{R,1} \end{aligned} \quad (120)$$

and

$$\begin{aligned} x_{t_r+m_{R-r-1}+1+k} &= q^{-2k} x_{t_r+1} = q^{-2(t_r-m_R+m_{R-r-1}+k)} x_1 \quad \text{for } 1 \leq k < n_{R-r-2}, \quad 0 \leq r \leq R-3 \\ x_{m_R+1+k} &= q^{-2k} x_1 \quad \text{for } 1 \leq k < n_{R-1} \end{aligned} \quad (121)$$

Here the first (second) line of (121) is omitted for $n_{R-r-2} = 1$ ($n_{R-1} = 1$). Similarly, the last line of (120) is dropped for $m_R = 0$. The level-zero module for the border strip is contained in the tensor product of evaluation modules

$$W_{x_1} \otimes W_{x_2} \otimes \dots \otimes W_{x_M} \quad (122)$$

$U_q(\widehat{gl}(2|2))$ acts on (122) via the iterated coproduct $\Delta^{(M-1)}(a)$ introduced by $\Delta^{(m)}(a) = (\Delta \otimes \mathbf{1}) \Delta^{(m-1)}(a)$ with $\Delta^{(1)}(a) = \Delta(a)$ and equations (19), (23). For x_k related to x_1 as specified by (119)-(121), the tensor product (122) contains an element $\nu_{\{n_i; m_i\}}$ characterized by the properties

$$\Delta^{(M-1)}(E_s^{l,+}) \nu_{\{n_i; m_i\}} = 0 \quad \text{for } s \in \mathbb{Z}, l = 1, 2, 3 \quad (123)$$

$$\begin{aligned} \Delta^{(M-1)}(h_1) \nu_{\{n_i; m_i\}} &= M \nu_{\{n_i; m_i\}} & \Delta^{(M-1)}(h_2) \nu_{\{n_i; m_i\}} &= -K \nu_{\{n_i; m_i\}} \\ \Delta^{(M-1)}(h_3) \nu_{\{n_i; m_i\}} &= 0 & \Delta^{(M-1)}(h_4) \nu_{\{n_i; m_i\}} &= (M-K) \nu_{\{n_i; m_i\}} \end{aligned} \quad (124)$$

and

$$\begin{aligned} \Delta^{(M-1)}(H_s^1) \nu_{\{n_i; m_i\}} &= q^s \frac{[s]}{s} \sum_{k=1}^M x_k^s \nu_{\{n_i; m_i\}} \\ \Delta^{(M-1)}(H_s^2) \nu_{\{n_i; m_i\}} &= \begin{cases} 0 & \text{if } R = 1 \\ -\frac{[s]}{s} \sum_{k \in \Upsilon_{\{n_i; m_i\}}} x_k^s \nu_{\{n_i; m_i\}} & \text{if } R > 1 \end{cases} \\ \Delta^{(M-1)}(H_s^3) \nu_{\{n_i; m_i\}} &= 0 \end{aligned} \quad (125)$$

$\forall s \neq 0$.

$\Upsilon_{\{n_i; m_i\}}$ denotes a subset of all $\{k \in \mathbb{N} \mid 1 \leq k \leq M\}$ defined for all $\{n_i; m_i\}$ with $R > 1$. It contains the value $k = 1$ for any of these sets $\{n_i; m_i\}$. In addition, the value $k > 1$ is included if the k -th box is occupied by the number 1 in the top labeling except for the uppermost of these boxes. All other values are not contained in $\Upsilon_{\{n_i; m_i\}}$. $\Upsilon_{\{4,2;2,3,4\}}$ for the example illustrated in Fig. 3 is given by $\{1, 6, 7, 11, 12, 13\}$. As a second example, for the border strip consisting of a single column of $n_1 + 1$ boxes the set $\Upsilon_{\{n_1; m_1=1, m_2=0\}} = \{1, 2, \dots, n_1\}$ results. According to (124), w_0 and w_1 occur $(M-K)$ -times and K -times in $\nu_{\{n_i; m_i\}}$, respectively. This implies the last of equations (125). Clearly, $\nu_{\{n_i; m_i\}}$ corresponds to the top labeling A of the border strip. The coproduct of $(H_{\pm s}^1 + H_{\pm s}^3)$ takes the simple form

$$\Delta(H_{\pm s}^1 + H_{\pm s}^3) = (H_{\pm s}^1 + H_{\pm s}^3) \otimes q^{-\frac{1}{2}sc(1 \mp 2)} + q^{\frac{1}{2}sc(1 \pm 2)} \otimes (H_{\pm s}^1 + H_{\pm s}^3) \quad s > 0 \quad (126)$$

where c denotes the level. Then the first of equations (125) follows from the third and

$$H_s^l(w_j \otimes x^{s'}) = (\delta_{l,j} + \delta_{l,j+1}) \cdot (-1)^{l-1} q^{s(1+\delta_{l,2})} \frac{[s]}{s} (w_j \otimes x^{s+s'}) \quad (127)$$

An evaluation making use of (119)-(121) leads to

$$\Delta^{(M-1)}(H_s^1) \nu_{\{n_i; m_i\}} = q^{2s(m_R - \delta_{R,1} + 1) - sM} \frac{[sM]}{s} x_1^s \nu_{\{n_i; m_i\}}$$

$$\Delta^{(M-1)}(H_s^2) \nu_{\{n_i; m_i\}} = -\frac{1}{s} x_1^s \left\{ \sum_{i=1}^{R-1} q^{-2s(t_{i-1} - m_R) + s(n_{R-i} + 1)} [sn_{R-i}] \right\} \nu_{\{n_i; m_i\}} \quad (128)$$

Equation (123) and the second of (125) result from a straightforward analysis employing the coproduct structure given in [42]. $\Delta^{(M-1)}(E_s^{1,-}) \nu_{\{n_i; m_i\}}$ and $\Delta^{(M-1)}(E_{s'}^{1,-}) \nu_{\{n_i; m_i\}}$ differ only by a complex number for any s, s' . In contrast, there are $R-1$ linear independent expressions $\Delta^{(M-1)}(E_s^{2,-}) \nu_{\{n_i; m_i\}}$.

The action of $U_q(\widehat{sl}(2|2))$ on $\nu_{\{n_i; m_i\}}$ gives rise to a finite-dimensional irreducible module denoted by $W_{x_1, \{n_i; m_i\}}$ for $R > 1$ and by $W_{x_1, \{M\}}$ for $R = 1$. Below, reference to a general parameter set $\{n_i; m_i\}$ will also include $\{M\}$. In analogy to the terminology used in [52], the eigenvalues of $\Delta^{(M-1)}(h_l)$ and $\Delta^{(M-1)}(H_s^l)$ given in (124) and (125) may be called the highest weight of $W_{x_1, \{n_i; m_i\}}$. The $U_q(gl(2|2))$ -characters of $W_{x_1, \{n_i; m_i\}}$ can be inferred from the Schur functions $s_{\lambda \setminus \mu}(\mathbf{p})$ via

$$\begin{aligned} ch_{\{n_i; m_i\}}(\mathbf{p}) &= \\ &= tr_{W_{x_1, \{n_i; m_i\}}} p_0^{\frac{1}{2}\Delta^{(M-1)}(h_1+h_2+h_3+h_4)} p_1^{\frac{1}{2}\Delta^{(M-1)}(h_1-h_2-h_3-h_4)} p_2^{-\frac{1}{2}\Delta^{(M-1)}(h_1+h_2-h_3-h_4)} p_3^{\frac{1}{2}\Delta^{(M-1)}(h_1+h_2+h_3-h_4)} \\ &= s_{\lambda \setminus \mu}(\mathbf{p}) \end{aligned} \quad (129)$$

Instead of $W_{x_1, \{n_i; m_i\}}$, a level-zero module contained in the tensor product of evaluation modules

$$W_{\bar{x}_1}^* \otimes W_{\bar{x}_2}^* \otimes \dots \otimes W_{\bar{x}_M}^* \quad (130)$$

may be associated to the border strip. To any border strip with parameters $R, \{n_i; m_i\}$ there corresponds a reversed border strip with the same number of rows and columns and $\{\bar{n}_i; \bar{m}_i\}$ given by $\bar{n}_i = n_{R-i}$ for $1 \leq i < R$ and $\bar{m}_i = m_{R-i+1} - \delta_{i,R} + \delta_{i,1}$ for $1 \leq i \leq R$. Both the labeling and the spectral parameters for the border strip $\{n_i; m_i\}$ are mapped onto (a second type of) labeling and a set of spectral parameters for the reversed border strip in two steps. First, the number as well as the spectral parameter in the k -th box of the border strip $\{n_i; m_i\}$ are attributed to the $M-k$ -th box of the reversed border strip $\{\bar{n}_i; \bar{m}_i\}$ for $1 \leq k \leq M$. In both stripes, the counting proceeds from the lowest leftmost box to the upper rightmost box. Then the mapping is completed by substituting each number l in the reversed border strip by $3-l$, $l = 0, 1, 2, 3$. Thus a row obtains each of the numbers 0 and 2 at most once and a column is occupied at most once by each of the numbers 1 and 3. The first rule of the type A labeling for the numbering is preserved. Below, this labeling will be referred to as type B labeling. For the top labeling B , the number 2 is attributed to the n_i lower boxes of a column with $n_i + 1 > 1$ boxes. All other boxes are occupied by the number 3. The spectral parameter in the k -th box is called \bar{x}_k . Fig. 4 illustrates the mapping for the border strip with $R = 3, m_1 = m_2 = m_3 = 2$ and $n_1 = 3, n_2 = 1$.

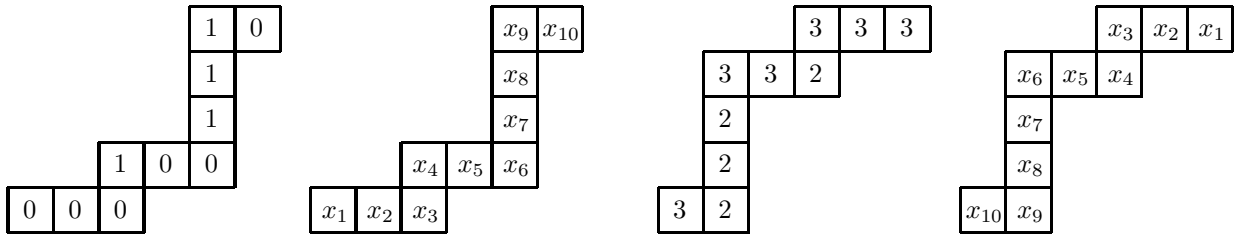


Fig. 4: Top labelling A and spectral parameters for $R = 3, m_1 = m_2 = m_3 = 2, n_1 = 3, n_2 = 1$ (left) and top labelling B for $\bar{m}_1 = 3, \bar{m}_2 = 2, \bar{m}_3 = 1, \bar{n}_1 = 3, \bar{n}_2 = 1$ and spectral parameters $x_i = \bar{x}_{10-i}$ for the reversed border strip (right)

A border strip with parameters R, M and $\{\bar{n}_i; \bar{m}_i\}$, type B labeling and spectral parameters \bar{x}_k is related to a level-zero module contained in (130). The replacements $x_k \rightarrow \bar{x}_k$ for $1 \leq k \leq M$ and $q \rightarrow q^{-1}$ in (119)-(121) provide the dependence among the spectral parameters in (130). With these relations, the tensor product (130) has an element $\bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ satisfying

$$\Delta^{(M-1)}(E_s^{l,+}) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} = 0 \quad \text{for } s \in \mathbb{Z}, l = 1, 2, 3 \quad (131)$$

$$\begin{aligned} \Delta^{(M-1)}(h_1) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= 0 & \Delta^{(M-1)}(h_2) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= K \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \\ \Delta^{(M-1)}(h_3) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= -M \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} & \Delta^{(M-1)}(h_4) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= (M - K) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \end{aligned} \quad (132)$$

and

$$\begin{aligned} \Delta^{(M-1)}(H_s^1) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= 0 \\ \Delta^{(M-1)}(H_s^2) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= \begin{cases} 0 & \text{if } R = 1 \\ \frac{[s]}{s} \sum_{k \in \bar{\Upsilon}_{\{\bar{n}_i; \bar{m}_i\}}} \bar{x}_k^s \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} & \text{if } R > 1 \end{cases} \\ \Delta^{(M-1)}(H_s^3) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= -q^s \frac{[s]}{s} \sum_{k=1}^M \bar{x}_k^s \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \end{aligned} \quad (133)$$

$\forall s \neq 0$. $\bar{\Upsilon}_{\{\bar{n}_i; \bar{m}_i\}}$ contains the value $k = M$ for any $\{\bar{n}_i; \bar{m}_i\}$ with $R > 1$ as well as each value $k < M$ provided that the k -th box is occupied by the number 2 in the top labeling B except for the lowest of these. The relations among the spectral parameters yield

$$\begin{aligned} \Delta^{(M-1)}(H_s^3) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= -q^{-2s(\bar{m}_R - \delta_{R,1}) + sM} \frac{[sM]}{s} \bar{x}_1^s \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \\ \Delta^{(M-1)}(H_s^2) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} &= \frac{[s]}{s} \bar{x}_1^s \left\{ \sum_{i=1}^{R-1} q^{2s(t_{R-1-i} - \bar{m}_R) - s(\bar{n}_i - 1)} \frac{[s\bar{n}_i]}{[s]} \right\} \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \end{aligned} \quad (134)$$

There exist $R - 1$ linear independent expressions $\Delta^{(M-1)}(E_s^{2,-}) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ while $\Delta^{(M-1)}(E_s^{3,-}) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ and $\Delta^{(M-1)}(E_{s'}^{3,-}) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ are linear dependant for any s, s' . In the following, the module generated by the action of $U'_q(\widehat{sl}(2|2))$ on $\bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ will be denoted by $W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ for $\bar{R} > 1$ and by $W_{\bar{x}_1, \{\bar{M}\}}^*$ for $\bar{R} = 1$.

Each labeling subject to the rules given below (130) can be mapped onto a component $(\dots, j_5^*, j_3^*, j_1^*)$ in $\{\tau^*\}_{\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}}$ with $j_{2r+1}^* = 3 \forall r \geq M$. Here the number in the k -th box is associated with $j_{2(M-k)+1}^*$. The value of $\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$ indicates the contribution of this component to the diagonal element of the CTM-Hamiltonian. In terms of the parameters of the border stripes it is expressed by

$$\bar{n}_{\{\bar{n}_i; \bar{m}_i\}} = - \sum_{i=1} \bar{n}_i \left\{ -\frac{1}{2}(\bar{n}_i + 1) + \sum_{j=1}^i (\bar{n}_j + \bar{m}_j) \right\} \quad (135)$$

In contrast to the correspondence between border stripes and configurations in $\{\tau\}_{n_{\{n_i; m_i\}}}$ described above, labelings of different border stripes may correspond to the same configuration $(\dots \otimes w_{j_5}^* \otimes w_{j_3}^* \otimes w_{j_1}^*)$. For example, all border stripes given by a single row with \bar{m}_1 boxes admit the top labeling B with the number 3 in each box. Thus they are mapped on the configuration $(\dots \otimes w_3^* \otimes w_3^* \otimes w_3^*)$ for arbitrary \bar{m}_1 .

VII. BORDER STRIPES AND THE LEVEL-ONE MODULE $V(\Lambda_0)$

To pursue further the relations between border stripes, level-one modules and the space of states investigated in Sect. III and IV, tensor products of $W_{x_1, \{n_i; m_i\}}$ and $W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ with independent sets of $\{n_i; m_i\}$ and $\{\bar{n}_i; \bar{m}_i\}$ need to be considered. Any two parameterizations $\{n_i; m_i\}$ and $\{\bar{n}_i; \bar{m}_i\}$ of border stripes with $n_i > 0, m_i > 0$ for $1 \leq i < R$ and $\bar{n}_i > 0, \bar{m}_i > 0$ for $1 \leq i < \bar{R}$ may be selected with the properties

$$M = \sum_{i=1}^R (n_i + m_i) = \sum_{i=1}^{\bar{R}} (\bar{n}_i + \bar{m}_i) \quad (136)$$

and

$$K = \sum_{i=1}^{R-1} n_i \geq \bar{K} = \sum_{i=1}^{\bar{R}-1} \bar{n}_i \quad (137)$$

A generator $a \in U'_q(\widehat{sl}(2|2))$ acts on $W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ via the coproduct $\Delta^{(2M-1)}(a)$ and equations (19), (23). For a more compact notation, explicit reference to the spectral parameters in (23) is suppressed in the remainder. The collection of all vectors in $W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ obtained by the action of $U'_q(\widehat{sl}(2|2))$ on $\nu_{\{n_i; m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ may be denoted by $W_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$. An element $\sigma \in W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ is called a highest weight vector if it is an eigenvector of $\{h_j, H_s^j\}$ and satisfies $E_s^{j,+} \sigma = 0 \forall s$ with $j = 1, 2, 3$.

Due to the first equation and (124), (132), the eigenvalue of $\Delta^{(2M-1)}(h_1 + h_3)$ vanishes on the tensor product

$$W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^* \quad (138)$$

In particular, if

$$\bar{x}_1 = q^{-2(M-1)+2(m_R+\bar{m}_R-\delta_{R,1}-\delta_{\bar{R},1})} x_1 \quad (139)$$

the linear combination $\Delta^{(2M-1)}(H_s^1 + H_s^3)$ annihilates all elements of (138) for any $s \neq 0$ as a consequence of (127), (125) and (133). Moreover, with (139) the spectral parameters x_k, \bar{x}_k take values in the set

$$q^{2(m_R-\delta_{R,1})} x_1, q^{2(m_R-\delta_{R,1}-1)} x_1, \dots, q^{2(m_R-\delta_{R,1}-M+1)} x_1 \quad (140)$$

Since $x_k \neq x_{k'}$ and $\bar{x}_k \neq \bar{x}_{k'}$ if $k \neq k'$, there exists a \bar{k} for any k such that $x_k = \bar{x}_{\bar{k}}$. The following list collects all combinations of $\{n_i; m_i\}$ and $\{\bar{n}_i; \bar{m}_i\}$ with $n_{\{n_i; m_i\}} + \bar{n}_{\{\bar{n}_i; \bar{m}_i\}} > -4$ and the properties (136), (137):

$R \ \bar{R}$	$\{n_i; m_i\}$	$\{\bar{n}_i; \bar{m}_i\}$	$n_{\{n_i; m_i\}}$	$\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$	$(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)_{h.w.}$
1 1	$m_1 = 1$	$\bar{m}_1 = 1$	-1	0	$(1, 0, -1, 2)$
2 1	$m_1 = 1, m_2 = 0, n_1 = 1$	$\bar{m}_1 = 2$	-2	0	$(2, -1, -2, 3)$
2 1	$m_1 = 1, m_2 = 0, n_1 = 2$	$\bar{m}_1 = 3$	-3	0	$(3, -2, -3, 4)$
1 1	$m_1 = 2$	$\bar{m}_1 = 2$	-3	0	$(2, 0, -2, 4)$
2 2	$m_1 = 1, m_2 = 0, n_1 = 1$	$\bar{m}_1 = 1, \bar{m}_2 = 0, \bar{n}_1 = 1$	-2	-1	$(2, 0, -2, 2)$

Here the rightmost column specifies the $U_q(gl(2|2))$ -weight of $\nu_{\{n_i; m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$. According to (124), (132) this is given by $(M, \bar{K} - K, -M, 2M - K - \bar{K})$. The remaining part of the highest weight follows from

$$\Delta^{(2M-1)}(H_s^l) (\nu_{\{n_i; m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}) = \Delta^{(M-1)}(H_s^l) \nu_{\{n_i; m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} + \nu_{\{n_i; m_i\}} \otimes \Delta^{(M-1)}(H_s^l) \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}} \quad (141)$$

for $l = 1, 2, 3$ and (125), (133). In general, the module $W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ with x_1 and \bar{x}_1 related by (139) is reducible. This is easily demonstrated for the simplest case $W_{x_1} \otimes W_{\bar{x}_1}^*$ with $x_1 = \bar{x}_1$. The highest weight vector conditions are satisfied by the two vectors

$$w_0 \otimes w_3^* \quad \text{and} \quad \sigma_0 = \sum_{i=1}^3 w_i \otimes w_i^* = q^{-1} \Delta(f_3 f_2 f_1 - f_1 f_2 f_3) w_0 \otimes w_3^* \quad (142)$$

At $x_1 = \bar{x}_1$, acting with the coproduct of the $U'_q(\widehat{sl}(2|2))$ -generators on $w_0 \otimes w_3^*$ does not produce the vector $w_0 \otimes w_0^* - w_1 \otimes w_1^* + w_2 \otimes w_2^* - w_3 \otimes w_3^*$. In turn, action of $U'_q(\widehat{sl}(2|2))$ on the latter gives rise to all $w_l \otimes w_{l'}^*$ with $0 \leq l, l' \leq 3$. Taking the quotient of all vectors emerging under the action of $U'_q(\widehat{sl}(2|2))$ on $w_0 \otimes w_3^*$ by the vector σ_0 provides an irreducible level-zero module $W_{x_1, \{1\}, x_1, \{1\}}$. The vector σ_0 may be seen as the first example of a particular type of highest weight vector occurring in each tensor product $W_{x_1, \{n_i; m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}^*$ with $R = 2, \bar{R} = 1, m_1 = 1, m_2 = 0$. These values correspond to border stripes given by the single column of $M = K + 1 = n_1 + 1$ boxes for the W -part

and the single row of M boxes for the W^* -part. Their spectral parameters are related by $x_k = \bar{x}_k = q^{-2(k-1)}x_1$ for $1 \leq k \leq K+1$. The highest weight vector properties are satisfied by

$$\nu_{\{K;1,0\}} \otimes \bar{\nu}_{\{K+1\}} = \left\{ \sum_{k=1}^K (-1)^k q^{-k} w_1^{\otimes(K-k)} \otimes w_0 \otimes w_1^{\otimes k} \right\} \otimes w_3^{\otimes(K+1)} \quad (143)$$

with $U_q(gl(2|2))$ -weight $(K+1, -K, -K-1, K+2)$ and by

$$\begin{aligned} \sigma_K &= \Delta^{(2M-1)}(f_3 f_2 f_1 - f_1 f_2 f_3) \cdot \\ &\cdot \Delta^{(2M-1)} \left\{ (e_1 e_0)^K + \sum_{k=1}^K A_k (f_3 f_2)^k (e_1 e_0)^{K-k} + \sum_{k=1}^K B_k (f_2 f_3)^k (e_1 e_0)^{K-k} \right\} (\nu_{\{K;1,0\}} \otimes \bar{\nu}_{\{K+1\}}), \\ A_k &= (-1)^k x_1^k q^{-k(K-1)} \frac{[K][K-1] \dots [K-k+1]}{[k][k-1] \dots [1]} \quad B_k = \frac{k+1}{[k+1]} A_k \end{aligned} \quad (144)$$

with $U_q(gl(2|2))$ -weight $(0, 0, 0, 0)$. The expression in the large parenthesis in (144) is annihilated by $\Delta^{(2M-1)}(e_j)$ with $j = 1, 2, 3$ but not by $\Delta^{(2M-1)}(E_s^{j,+})$ with general s . To include the value $K = 0$, it should be replaced by the unit. σ_K is a one-dimensional $U'_q(\widehat{sl}(2|2))$ -module. With the spectral parameters specified above, no further elements with highest weight vector properties are found in $W_{x_1, \{K;1,0\}, x_1, \{K+1\}}$. For any K , the quotient $W_{x_1, \{K;1,0\}, x_1, \{K+1\}} / \sigma_K$ forms an irreducible level-zero module of $U'_q(\widehat{sl}(2|2))$. Removing σ_K corresponds to elimination of a vector of weight $(0, 0, 0, 0)$ in the level-one module $V(\Lambda_0)$ at grade $-K-1$. This is done by imposing the condition $(H_{-K-1}^1 + H_{-K-1}^3) \hat{\lambda}_0 = 0$. Here $\hat{\lambda}_0$ denotes the highest weight vector of $V(\Lambda_0)$ with $h_j \hat{\lambda}_0 = \delta_{j,0} \hat{\lambda}_0$, $0 \leq j \leq 4$.

In the remaining two cases listed in the above table, application of the $U'_q(\widehat{sl}(2|2))$ -generators on $\nu_{\{n_i, m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i, \bar{m}_i\}}$ produces only a part of $W_{x_1, \{n_i, m_i\}} \otimes W_{\bar{x}_1, \{\bar{n}_i, \bar{m}_i\}}^*$. If $R = \bar{R} = 1$, equation (139) leads to $x_k = q^{2(k-1)}x_1$, $\bar{x}_k = q^{2(M-k)}x_1$ with $1 \leq k \leq M$. With these relations,

$$\Delta^{(2M-1)}(x_1^{-1} e_0 - f_3 f_2 f_1 + f_1 f_2 f_3 + q f_2 f_1 f_3) (\nu_{\{M\}} \otimes \bar{\nu}_{\{M\}}) = 0 \quad M = 1, 2, 3, \dots \quad (145)$$

A highest weight vector with $U_q(gl(2|2))$ -weight $(M-1, 0, -M+1, 2M-2)$ is found in $W_{x_1, \{M\}} \otimes W_{q^{2(M-1)}x_1, \{M\}}^*$ but not present in $W_{x_1, \{M\}, q^{2(M-1)}x_1, \{M\}}$. For $M = 2$ it is given by

$$\sum_{l=0}^3 (-1)^{|l|} w_0 \otimes w_l \otimes w_l^* \otimes w_3^* - q^{-1} \sum_{l=0}^3 q^{-\delta_{l,0} - \delta_{l,3}} w_l \otimes w_0 \otimes w_3^* \otimes w_l^* \quad (146)$$

Furthermore, the tensor product $W_{x_1, \{2\}} \otimes W_{q^2 x_1, \{2\}}^*$ contains the vector $\tilde{\sigma} = (w_3 \otimes w_1 - q w_1 \otimes w_3) \otimes (w_2^* \otimes w_0^* - q w_0^* \otimes w_2^*)$ with $U_q(gl(2|2))$ -weight $(0, 0, 0, -2)$ which is not reached by the action of $U_q(\widehat{sl}(2|2))$ on $\nu_{\{2\}} \otimes \bar{\nu}_{\{2\}}$ or on (146). Application of $U'_q(\widehat{sl}(2|2))$ -generators on $\tilde{\sigma}$ leads to (146) but not to $\nu_{\{2\}} \otimes \bar{\nu}_{\{2\}}$. As a result, sixteen elements of $W_{x_1, \{2\}} \otimes W_{q^2 x_1, \{2\}}^*$ are not found in $W_{x_1, \{2\}, q^2 x_1, \{2\}}$ which contains 48 different vectors.

For $R = \bar{R} = 2$ and $m_1 - 1 = m_2 = \bar{m}_1 - 1 = \bar{m}_2 = 0$, the spectral parameters are related by $x_k = q^{-2(k-1)}x_1$ and $\bar{x}_k = q^{-2(M-k)}x_1$ with $1 \leq k \leq M$. Then

$$\Delta^{(2M-1)}(x_1^{-1} e_3 e_1 e_0 - q f_2) (\nu_{\{K;0,0\}} \otimes \bar{\nu}_{\{K;0,0\}}) = 0 \quad K = 1, 2, 3, \dots \quad (147)$$

and $W_{x_1, \{K;1,0\}} \otimes W_{q^{-2K}x_1, \{K;1,0\}}^*$ has a vector with $U_q(gl(2|2))$ -weight $(K, 0, -K, 2)$ not found in $W_{x_1, \{K;1,0\}, q^{-2K}x_1, \{K;1,0\}}$. Though it is annihilated by $\Delta^{(2K+1)}(e_j)$ with $j = 1, 2, 3$, it does not satisfy the highest weight vector properties. At $K = 1$ it is given by

$$(w_0 \otimes w_1 - q w_1 \otimes w_0) \otimes (w_1^* \otimes w_3^* - q w_3^* \otimes w_1^*) - (w_0 \otimes w_2 - q w_2 \otimes w_0) \otimes (w_2^* \otimes w_3^* + q w_3^* \otimes w_2^*) \quad (148)$$

In addition, $W_{x_1, \{1;1,0\}} \otimes W_{q^{-2}x_1, \{1;1,0\}}^*$ contains the vector $\hat{\sigma} = (w_0 \otimes w_2 - q w_2 \otimes w_0) \otimes (w_1^* \otimes w_3^* - q w_3^* \otimes w_1^*)$ which does not result from the action of $U'_q(\widehat{sl}(2|2))$ on (148) or $\nu_{\{1;1,0\}} \otimes \bar{\nu}_{\{1;1,0\}}$. Action of the $U'_q(\widehat{sl}(2|2))$ -generators on $\hat{\sigma}$ creates the whole tensor product $W_{x_1, \{1;1,0\}} \otimes W_{q^{-2}x_1, \{1;1,0\}}^*$. Sixteen vectors are present in $W_{x_1, \{1;1,0\}} \otimes W_{q^{-2}x_1, \{1;1,0\}}^*$

but not in $W_{x_1, \{1; 1, 0\}, q^{-2}x_1, \{1; 1, 0\}}$ which has 48 different vectors. No highest weight vectors besides $\nu_{\{2\}} \otimes \bar{\nu}_{\{2\}}$ and $\nu_{\{2; 1, 0\}} \otimes \bar{\nu}_{\{2; 1, 0\}}$ are found in $W_{x_1, \{2\}, q^2x_1, \{2\}}$ and $W_{x_1, \{1; 1, 0\}, q^{-2}x_1, \{1; 1, 0\}}$, respectively.

The quotient of $W_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ by all highest weight vectors $\rho_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}} \in W_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ different from $\nu_{\{n_i; m_i\}} \otimes \bar{\nu}_{\{\bar{n}_i; \bar{m}_i\}}$ and the vectors obtained by the action of $U'_q(\widehat{sl}(2|2))$ on $\rho_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ may be called $\hat{W}_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$. It is easily verified that for $n_{\{n_i; m_i\}} + \bar{n}_{\{\bar{n}_i; \bar{m}_i\}} \geq -3$ the $U_q(\widehat{gl}(2|2))$ -weights of the vectors present in $\hat{W}_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ are in one-to-one correspondence with the $U_q(\widehat{gl}(2|2))$ -weights of the vectors found in the level-one $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module $V(\Lambda_0)$ at grade $n_{\{n_i; m_i\}} + \bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$. This result may be assumed to hold true at any grade.

Conjecture II: The character of the $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module $V(\Lambda_0)$ at level one can be written

$$ch_{V(\Lambda_0)}(\rho, \mathbf{p}) = 1 + \sum'_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \rho^{n_{\{n_i; m_i\}} + \bar{n}_{\{\bar{n}_i; \bar{m}_i\}}} ch_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}(\mathbf{p}) \quad (149)$$

Here \sum' denotes the sum over all $\{n_i; m_i\}$ and $\{\bar{n}_i; \bar{m}_i\}$ with $\sum_i (n_i + m_i) = \sum_i (\bar{n}_i + \bar{m}_i)$ and $\sum_i n_i \geq \sum_i \bar{n}_i$. The character on the rhs of (149) is defined by

$$ch_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \equiv tr_{\hat{W}_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}} p_0^{\frac{1}{2}\Delta^{(2M-1)}(h_1+h_2+h_3+h_4)} p_1^{\frac{1}{2}\Delta^{(2M-1)}(h_1-h_2-h_3-h_4)} \cdot p_2^{-\frac{1}{2}\Delta^{(2M-1)}(h_1+h_2-h_3-h_4)} p_3^{\frac{1}{2}\Delta^{(2M-1)}(h_1+h_2+h_3-h_4)} \quad (150)$$

with x_1 and \bar{x}_1 related by (139). Its general evaluation requires further investigation. The submodules $\hat{W}_{x_1, \{n_i; m_i\}, \bar{x}_1, \{\bar{n}_i; \bar{m}_i\}}$ are special cases of the modules introduced in the last section of [53]. Apparently, their structure has not been studied so far. In this context, it may be helpful to observe that the $U_q(\widehat{gl}(2|2))$ -characters in [37] are written as sums of products of supersymmetric (Hook) Schur functions. Multiplying (149) by the factor $\prod_{t=1}^{\infty} (1 - q^t)^{-2}$ gives an expression for the $U_q(\widehat{gl}(2|2))$ -character of $V(\Lambda_0)$. It is easily checked that at grades ≥ -3 the expression reproduces the character formula 3.14 in [35] with $s = 0$. Combinatorial identities can be obtained from comparison with the results in [37].

VIII. BORDER STRIPES AND THE MODULE $\tilde{V}(\Lambda_0)$

A. Infinite border stripes

A one-to-one mapping between the complete space of configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ and the semi-standard super tableaux of border stripes can be achieved by identifying the type B labelings of infinite border stripes with the components $(\dots \otimes w_{j_5}^* \otimes w_{j_3}^* \otimes w_{j_1}^*)$. To this aim, the infinite border strip parameterized by \bar{R} , $\{\bar{n}_i, \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = 0$ is introduced as the limit $\bar{m}_{\bar{R}} \rightarrow \infty$ of the border stripes characterized by fixed values of $\bar{R} > 1$ and (finite) \bar{n}_i, \bar{m}_i , $1 \leq i \leq \bar{R} - 1$. For $\bar{R} = 1$, the infinite border strip parameterized by $\{1\}$ is a half-infinite row. The values of $\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$ in (135) to be equated with the diagonal elements of the CTM don't depend on the value of $\bar{m}_{\bar{R}}$. In contrast to the infinite border stripes considered for $U_q(\widehat{sl}(N))$ -models in [16], each border strip \bar{R} , $\{\bar{n}_i; \bar{m}_i\}$ accommodates an infinite number of semi-standard super tableaux. Due to the boundary condition imposed on the configurations, almost all boxes are occupied by the number 3. For $l = 0, 1, 2$ the multiplicity of the number l in a given type B labeling T_B may be denoted by $N_l(T_B)$. The values of $N_0(T_B)$ and $N_2(T_B)$ are bound from above for a given set of parameters \bar{R} , \bar{n}_i, \bar{m}_i , $1 \leq i \leq \bar{R} - 1$. Hence a skew Schur function $\tilde{s}_{\lambda \setminus \mu}(\tilde{\mathbf{p}})$ can be introduced as

$$\tilde{s}_{\lambda \setminus \mu}(\tilde{\mathbf{p}}) \equiv \sum_{T_B \in T_B(\lambda \setminus \mu)} \tilde{p}_0^{N_0(T_B)} \tilde{p}_1^{N_1(T_B)} \tilde{p}_2^{N_2(T_B)} \quad (151)$$

with $|\tilde{p}_1| < 1$. Here the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\bar{K}+1})$ and $\mu = (\mu_1, \mu_2, \dots, \mu_{\bar{K}+1})$ satisfy $\lambda_{\bar{K}+1} = 1$ and $\mu_{\bar{K}+1} = 0$. The remaining values $(\lambda_1, \lambda_2, \dots, \lambda_{\bar{K}}), (\mu_1, \mu_2, \dots, \mu_{\bar{K}})$ are related to the sets $\{\bar{n}_i\}_{1 \leq i \leq \bar{R}-1}$ and $\{\bar{m}_i\}_{1 \leq i \leq \bar{R}-1}$ by (109) with R, K, m_i, n_i replaced by $\bar{R}, \bar{K}, \bar{m}_i, \bar{n}_i$ for $1 \leq i \leq \bar{R} - 1$ and $m_{\bar{R}}$ set to zero. If $\bar{R} = 1$, the partitions are $\lambda = (1)$ and $\mu = (0)$. $T_B(\lambda \setminus \mu)$ denotes the set of all type B labelings of the infinite border strip characterized by \bar{R} , $\{\bar{n}_i, \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = \delta_{\bar{R}, 1}$. A determinant formula for $\tilde{s}_{\lambda \setminus \mu}(\tilde{\mathbf{p}})$ is written as

$$\tilde{s}_{\lambda \setminus \mu}(\tilde{\mathbf{p}}) = \det(\tilde{E})_{1 \leq i, j \leq \bar{K}+1} \quad (152)$$

where $\tilde{E}_{i,j} = \tilde{e}_{\lambda_i - \mu_j - i + j}$ for $i, j \leq \bar{K}$ with $\tilde{e}_0 = 1$, $\tilde{e}_{-m} = 0$ and

$$\tilde{e}_m = \tilde{p}_1^m + \tilde{p}_0 \tilde{p}_1^{m-1} + \frac{1 + \tilde{p}_2}{1 - \tilde{p}_1} (1 - \tilde{p}_1^m + \tilde{p}_0 - \tilde{p}_0 \tilde{p}_1^{m-1}) \quad (153)$$

for $m = 1, 2, 3, \dots$ and $\tilde{E}_{\bar{K}+1,i} = 0$ for $1 \leq i < \bar{K}$, $\tilde{E}_{\bar{K}+1,\bar{K}} = 1$, $\tilde{E}_{i,\bar{K}+1} = (1 - \tilde{p}_1)^{-1}(1 + \tilde{p}_0)(1 + \tilde{p}_2)$. For any type B labeling T_B of an infinite border strip, a $U_q(gl(2|2))$ -weight may be defined by

$$(-N_0(T_B) - N_1(T_B), N_1(T_B) + N_2(T_B), N_0(T_B) + N_1(T_B), -2N_0(T_B) - N_1(T_B) - N_2(T_B)) \quad (154)$$

The boxes in the infinite border strip may be counted starting from the upper rightmost box. Then the number in the k -th box is associated with the entry j_{2k-1}^* of the component $(\dots, j_5^*, j_3^*, j_1^*)$. The restricted type A labelings of finite border stripes are associated with the components $(\dots, j_6, j_4, j_2) \neq (\dots, 3, 3, 3)$ as specified in Sect. VI. Thus pairs of border stripes parameterized by $R, \{n_i; m_i\}$ and $\bar{R}, \{\bar{n}_i; \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$ allow to characterize the space of half-infinite configurations contained in $\{\tau\}_n \otimes \{\tau^*\}_{\bar{n}}$ with $n \neq 0$. A labeling $T_{A,B}$ of such a pair consists of a restricted type A labeling T'_A of the finite border strip $R, \{n_i; m_i\}$ and of a type B labeling T_B of the infinite border strip $\bar{R}, \{\bar{n}_i; \bar{m}_i\}$. The value $R = 0$ related to the configuration $(\dots, 3, 3, 3)$ may be included in the set of finite border stripes. In this case, the labelings $T_{A,B}$ are given by the type B labelings T_B of the infinite border strip. Clearly, the labelings $T_{A,B}$ are in one-to-one correspondence with the half-infinite configurations $(\dots, j_4, j_3^*, j_2, j_1^*)$ with $j_r = 3$ for almost all r . The number $n_{\{n_i; m_i\}} + \bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$ for $R > 0$ (or $\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}$ for $R = 0$) equals the diagonal element of the corner transfer matrix Hamiltonian acting on the corresponding configuration. The multiplicities of the numbers $l = 0, 1, 2$ in the restricted type A labeling T'_A may be denoted by $N_l(T'_A)$. Then a $U_q(gl(2|2))$ -weight of the labeling $T_{A,B}$ is introduced as the sum of (154) and

$$(N_0(T'_A) + N_1(T'_A), -N_1(T'_A) - N_2(T'_A), -N_0(T'_A) - N_1(T'_A), 2N_0(T'_A) + N_1(T'_A) + N_2(T'_A)) \quad (155)$$

If $R = 0$, (155) is replaced by $(0, 0, 0, 0)$.

This weight coincides with the $U_q(gl(2|2))$ -weight of the half-infinite configuration corresponding to the labeling $T_{A,B}$ provided that the former is evaluated with respect to the reference weight \bar{h}^A . The set of the $U_q(gl(2|2))$ -weights attributed to all labelings $T_{A,B}$ of a given pair of border stripes $R, \{n_i; m_i\}$ and $\bar{R}, \{\bar{n}_i; \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$ will be denoted by $\Xi_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$. For $R = 0$, the set of the weights (154) attributed to all labelings T_B is denoted by $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$. The simplest set $\Xi_{\{1\}}$ coincides with $\sigma(0, 0, 0, 0)$ introduced in (70).

Conjecture I and the correspondence between labelings $T_{A,B}$ and the half-infinite configurations imply a decomposition of the affine character of $\tilde{V}(\Lambda_0)$ in terms of skew Schur functions.

Conjecture III: A spectral decomposition for the character (74) of the $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -module $\tilde{V}(\Lambda_0)$ at level one is given by

$$ch_{\tilde{V}(\Lambda_0)}(\rho, p_0, p_1, p_2) = \left(1 + \sum_{\lambda \setminus \mu} \rho^{n_{\lambda \setminus \mu}} s'_{\lambda \setminus \mu}(p_0, p_1, p_2, 1)\right) \cdot \sum_{\bar{\lambda} \setminus \bar{\mu}} \rho^{\bar{n}_{\bar{\lambda} \setminus \bar{\mu}}} \tilde{s}_{\bar{\lambda} \setminus \bar{\mu}}(p_0^{-1}, p_1^{-1}, p_2^{-1}) \quad (156)$$

where $|p_1| > 1$. In (156), $\sum_{\lambda \setminus \mu}$ means the sum over all partitions related to finite border stripes $R, \{n_i; m_i\}$ with $R > 0$. $\sum_{\bar{\lambda} \setminus \bar{\mu}}$ is the sum over all partitions related to infinite border stripes $\bar{R}, \{\bar{n}_i; \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$. The values of $n_{\lambda \setminus \mu}$ and $\bar{n}_{\bar{\lambda} \setminus \bar{\mu}}$ are given by equations (113) and (135) with the parameters $\{n_i; m_i\}$ and $\{\bar{n}_i; \bar{m}_i\}$ associated to $\lambda \setminus \mu$ and $\bar{\lambda} \setminus \bar{\mu}$, respectively.

Infinite Young skew diagrams of the type described above and the associated characters or skew Schur functions have not been introduced so far. In the following section, another classification of the half-infinite configurations is proposed.

B. Infinite-dimensional $U_q(gl(2|2))$ -modules

The aim of this section is a description of the complete space Ω_A of configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ in terms of level-zero modules of $U_q(\widehat{sl}(2|2))$. These modules can be constructed from finite tensor products of both

finite- and infinite-dimensional evaluation modules. A suitable infinite-dimensional evaluation module is obtained from the $U'_q(\widehat{sl}(2|2))$ -module V with basis $\{v_{j,t}\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$ and

$$\begin{aligned}
h_0 v_{j,t} &= -(t+1 - \delta_{j,0}) v_{j,t} & h_1 v_{j,t} &= -(t+1 - \delta_{j,2}) v_{j,t} \\
h_2 v_{j,t} &= (t+1 - \delta_{j,0}) v_{j,t} & h_3 v_{j,t} &= (t+1 - \delta_{j,2}) v_{j,t} \\
f_3 v_{3,t} &= v_{2,t+1} & e_3 v_{2,t+1} &= [t+1] v_{3,t} \\
f_3 v_{0,t} &= -v_{1,t} & e_3 v_{1,t} &= -[t+1] v_{0,t} \\
f_2 v_{2,t} &= v_{3,t} & e_2 v_{3,t} &= [t+1] v_{2,t} \\
f_2 v_{1,t} &= -\frac{[t+1]}{[t+2]} v_{0,t+1} & e_2 v_{0,t+1} &= -[t+2] v_{1,t} \\
f_1 v_{3,t} &= v_{0,t} & e_1 v_{0,t} &= -[t+1] v_{3,t} \\
f_1 v_{2,t} &= v_{1,t-1} & e_1 v_{1,t} &= -[t+1] v_{2,t+1} \\
f_0 v_{1,t} &= q^{-2}[t+1]^2 v_{2,t} & e_0 v_{2,t} &= -q^2[t+1]^{-1} v_{1,t} \\
f_0 v_{0,t+1} &= q^{-2}[t+1][t+2] v_{3,t} & e_0 v_{3,t} &= -q^2[t+2]^{-1} v_{0,t+1}
\end{aligned} \tag{157}$$

A $U_q(gl(2|2))$ -weight matching the first choice in (55) for the weights of the configurations is given by

$$h_4 v_{j,t} = -(t+1 + \delta_{j,0} + 2\delta_{j,1}) v_{j,t} \tag{158}$$

On $V_z = V \otimes \mathbb{C}[z, z^{-1}]$ a $U_q(\widehat{sl}(2|2))$ -structure is defined by (23) with v_j replaced by $v_{j,t}$. Adding $(0,0,0,0)$ to the $U_q(gl(2|2))$ -weights of V_z yields the set $\Xi_{\{1\}}$. In terms of V_z , level-zero modules with the $U_q(gl(2|2))$ -weights given by the weight sets $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$ with $\bar{R} > 1$ can be specified. There are two type B labelings of the infinite border strip \bar{R} , $\{\bar{n}_i; \bar{m}_i\}$ attributing only the numbers 2 and 3 to the boxes. One of them coincides with the top labeling B for the finite border strip \bar{R} , $\{\bar{n}_i; \bar{m}_i\}$ in the first $\bar{M} = \sum_i (\bar{n}_i + \bar{m}_i)$ boxes (counted from the right end). All remaining boxes receive the number 3. The $U_q(gl(2|2))$ -weight (154) attributed to this labeling provides the highest weight for a level-zero module related to $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$. This weight is given by $(0, \bar{K}, 0, -\bar{K})$ with $\bar{K} = \sum_i \bar{n}_i$. Replacing the number 3 in the upper rightmost box by the number 2 yields the second labeling. Its $U_q(gl(2|2))$ -weight $(0, \bar{K}+1, 0, -\bar{K}-1)$ provides a highest weight for another level-zero module.

A simple example is provided by the infinite border strip $\bar{R} = 2, \{\bar{K}; 1, 0\}$. The weights of the set $\Xi_{\{\bar{K}; 1, 0\}}$ coincide with the weights of the two $U_q(gl(2|2))$ -modules with highest weights $(0, \bar{K}, 0, -\bar{K})$ and $(0, \bar{K}+1, 0, -\bar{K}-1)$. In terms of the evaluation modules V_z , the related level-zero $U_q(\widehat{sl}(2|2))$ -modules are expressed by

$$V_{y_1} \otimes V_{q^2 y_1} \otimes V_{q^4 y_1} \otimes \dots \otimes V_{q^{2(\bar{K}-1)} y_1} \quad \text{and} \quad V_{y_1} \otimes V_{q^2 y_1} \otimes V_{q^4 y_1} \otimes \dots \otimes V_{q^{2\bar{K}} y_1} \tag{159}$$

A second example is the infinite border strip $\bar{R} = 2, \{1; \bar{m}_1 > 1, 0\}$. The weights of the set $\Xi_{\{1; \bar{m}_1, 0\}}$ exactly match the weights of the $U_q(gl(2|2))$ -modules with highest weights $(0, 1, 0, -1)$ and $(-k, 2, k, -2)$ with $0 \leq k \leq \bar{m}_1 - 1$. The related level-zero $U_q(\widehat{sl}(2|2))$ -modules read

$$V_{y_1} \quad \text{and} \quad V_{y_1} \otimes V_{q^{2\bar{m}_1} y_1} \tag{160}$$

Generally, an arbitrary infinite border strip with $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$ corresponds to two irreducible level-zero $U_q(\widehat{sl}(2|2))$ -modules with highest weights

$$(0, \bar{K}, 0, -\bar{K}) \tag{161}$$

and

$$(0, \bar{K}+1, 0, -\bar{K}-1) \tag{162}$$

The weight (161) refers to a tensor product of \bar{K} evaluation modules V_z . Their spectral parameters are given by the spectral parameters \bar{x}_k with k contained in the set $\tilde{\Upsilon}_{\{\bar{n}_i; \bar{m}_i\}}$ in equation (133) multiplied by a factor q^{-2} . According to Sect. VI, these spectral parameters can be expressed as $q^{2r_j} \bar{x}_1$ for \bar{K} suitable values of r_j where $1 \leq r_j \leq \bar{M}$ and $r_j > r_{j'}$ if $j > j'$ and $k, k' \in \tilde{\Upsilon}_{\{\bar{n}_i; \bar{m}_i\}}$ for $q^{2r_j} \bar{x}_1 = \bar{x}_k$, $q^{2r_{j'}} \bar{x}_1 = \bar{x}_{k'}$. With the notation $y_j = q^{2(r_j-1)} \bar{x}_1$, the level-zero module can be written

$$V_{y_1} \otimes V_{y_2} \otimes V_{y_3} \otimes \dots \otimes V_{y_{\bar{K}}} \quad (163)$$

Similarly, the second level-zero module related to (162) is expressed by

$$V_{y_1} \otimes V_{y_2} \otimes V_{y_3} \otimes \dots \otimes V_{y_{\bar{K}}} \otimes V_{q^{2(\bar{M}-1)} y_1} \quad (164)$$

where $y_{\bar{K}} = q^{2(\bar{M}-\bar{m}_1-1)} y_1$ according to the previous section. For $\bar{M} = \bar{K} + 1 = 1$, the tensor product $V_{y_1} \otimes V_{y_2} \otimes \dots \otimes V_{y_{\bar{K}}}$ in (163) and (164) is replaced by the one-dimensional module with $U_q(gl(2|2))$ -weight $(0, 0, 0, 0)$. A tensor product $V_{z_1} \otimes V_{z_2} \otimes \dots \otimes V_{z_L}$ with arbitrary spectral parameters z_i and $L \geq 1$ may be denoted by $V_{\{z_i\}}$. Its highest weight vector $\tilde{\nu}_{\{z_i\}}$ satisfies

$$\begin{aligned} H_s^1 \tilde{\nu}_{\{z_i\}} &= 0 \\ H_s^2 \tilde{\nu}_{\{z_i\}} &= q^{2s} \frac{[s]}{s} \sum_{i=1}^L z_i^s \tilde{\nu}_{\{z_i\}} \\ H_s^3 \tilde{\nu}_{\{z_i\}} &= 0 \end{aligned} \quad (165)$$

Explicit analysis of the weight structure reveals that the $U_q(gl(2|2))$ -weights of (163) and (164) exactly coincide with the weights of the set $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$. Level-zero modules with the weight structure described by the sets $\Xi_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ are obtained in two steps. First, appropriate tensor products of M evaluation modules W_z^* with (163), (164) are considered. They contain level-zero modules with the $U_q(gl(2|2))$ -weights given by the sets

$$\begin{aligned} \Xi_{\{\bar{n}_i; \bar{m}_i\}}^{(M)} = \\ \left\{ (-N_0(T_B) - N_1(T_B), N_1(T_B) + N_2(T_B), -M + N_0(T_B) + N_1(T_B), M - 2N_0(T_B) - N_1(T_B) - N_2(T_B)) \right\} \end{aligned} \quad (166)$$

Here $M = \sum_i (n_i + m_i)$ and $N_l(T_B)$ is defined as in equation (154). Stated briefly, a uniform shift by $(0, 0, -M, M)$ applied on $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$ yields the set $\Xi_{\{\bar{n}_i; \bar{m}_i\}}^{(M)}$. A priori, several choices for the tensor products are possible. In the construction outlined below, all finite-dimensional modules $W_{\bar{x}_i, \{\bar{n}_i; \bar{m}_i\}}^*$ with $\bar{m}_{\bar{K}} = \delta_{\bar{K}, 1}$ involved in the decomposition of $V(\Lambda_0)$ in Sect. VII appear as submodules of suitable quotients. For $M < \bar{m}_1$, the level-zero module contains the whole tensor product

$$W_{q^{2\bar{m}_1} y_{\bar{K}}}^* \otimes W_{q^{2(\bar{m}_1-1)} y_{\bar{K}}}^* \otimes W_{q^{2(\bar{m}_1-2)} y_{\bar{K}}}^* \otimes \dots \otimes W_{q^{2(\bar{m}_1-M+1)} y_{\bar{K}}}^* \otimes V_{y_1} \otimes V_{y_2} \otimes \dots \otimes V_{y_{\bar{K}}} \quad (167)$$

If $M \geq \bar{m}_1$ and $\bar{M} > 1$, the tensor product (167) has a vector $\check{\nu}_{\{M; \bar{m}_1; y_i\}}$ with $U_q(gl(2|2))$ -weight $(-\bar{m}_1, \bar{K} + \bar{m}_1, -M + \bar{m}_1, M - \bar{K} - \bar{m}_1)$ and the highest weight properties

$$\Delta^{(M+\bar{K}-1)}(E_s^{l,+}) \check{\nu}_{\{M; \bar{m}_1; y_i\}} = 0 \quad \text{for } s \in \mathbb{Z}, l = 1, 2, 3 \quad (168)$$

$$\begin{aligned} \Delta^{(M+\bar{K}-1)}(H_s^1) \check{\nu}_{\{M; \bar{m}_1; y_i\}} &= -q^{s(2\bar{M}-\bar{m}_1)} \frac{[s\bar{m}_1]}{s} y_1^s \check{\nu}_{\{M; \bar{m}_1; y_i\}} \\ \Delta^{(M+\bar{K}-1)}(H_s^2) \check{\nu}_{\{M; \bar{m}_1; y_i\}} &= q^{2s} \frac{[s]}{s} \left\{ q^{s\bar{m}_1} \frac{[s(\bar{m}_1+1)]}{[s]} y_1^s + \sum_{i=1}^{\bar{K}-1} y_i^s \right\} \check{\nu}_{\{M; \bar{m}_1; y_i\}} \\ \Delta^{(M+\bar{K}-1)}(H_s^3) \check{\nu}_{\{M; \bar{m}_1; y_i\}} &= -q^{s(2\bar{M}-M-\bar{m}_1)} \frac{[s(M-\bar{m}_1)]}{s} y_1^s \check{\nu}_{\{M; \bar{m}_1; y_i\}} \end{aligned} \quad (169)$$

The level-zero module corresponding to the set $\Xi_{\bar{n}_i; \bar{m}_i}^{(M)}$ with $M \geq \bar{m}_1$, $\bar{M} > 1$ is the quotient $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ of the tensor product (167) by the module generated from $\tilde{V}_{\{M; \bar{m}_1; y_i\}}$ by the level-zero action of $U_q(\widehat{sl}(2|2))$. In the following, $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with $M < \bar{m}_1$ refers to the tensor product (167). $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ need not be irreducible. In particular, for $\bar{M} = M$ a highest weight vector is found at the $U_q(gl(2|2))$ -weight $(-M + \bar{K}, M, -\bar{K}, 0)$. Then the quotient of $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ by all vectors resulting from the action of $U_q(\widehat{sl}(2|2))$ on this vector includes a $U_q(\widehat{sl}(2|2))$ -submodule isomorphic to the finite-dimensional module $W_{q^{2(M-1)}y_1, \{\bar{n}_i; \bar{m}_i\}}^*$. This part arises from the action on the vector $w_3^{* \otimes M} \otimes v_{2,0}^{\otimes \bar{K}}$.

In addition to $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$, a part of the tensor product

$$W_{q^{2\bar{m}_1}y_{\bar{K}}}^* \otimes W_{q^{2(\bar{m}_1-1)}y_{\bar{K}}}^* \otimes W_{q^{2(\bar{m}_1-2)}y_{\bar{K}}}^* \otimes \dots \otimes W_{q^{2(\bar{m}_1-M+1)}y_{\bar{K}}}^* \otimes V_{y_1} \otimes V_{y_2} \otimes \dots \otimes V_{y_{\bar{K}}} \otimes V_{q^{2(M-1)}y_1} \quad (170)$$

is required in the case $M < \bar{m}_1$. The level-zero action of $U_q(\widehat{sl}(2|2))$ on the highest weight vector $w_3^{* \otimes M} \otimes v_{2,0}^{\otimes (\bar{K}+1)}$ in (170) does not exhaust the whole tensor product. The quotient $\hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ of the tensor product (170) by all vectors emerging from the action of $U_q(\widehat{sl}(2|2))$ on $w_3^{* \otimes M} \otimes v_{2,0}^{\otimes (\bar{K}+1)}$ contains a vector $\hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with $U_q(gl(2|2))$ -weight $(-M, \bar{K} + M + 1, 0, -\bar{K} - 1)$ and the properties

$$\Delta^{(M+\bar{K}-1)}(E_s^{l,+}) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} = 0 \quad \text{for } s \in \mathbb{Z}, l = 1, 2, 3 \quad (171)$$

$$\begin{aligned} \Delta^{(M+\bar{K}-1)}(H_s^1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} &= -q^{s(2\bar{M}-M)} \frac{[Ms]}{s} y_1^s \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \\ \Delta^{(M+\bar{K}-1)}(H_s^2) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} &= q^{2s} \frac{[s]}{s} \left\{ q^{s(2\bar{M}-M-2)} \frac{[s(M+1)]}{[s]} y_1^s + \sum_{i=1}^{\bar{K}} y_i^s \right\} \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \\ \Delta^{(M+\bar{K}-1)}(H_s^3) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} &= 0 \end{aligned} \quad (172)$$

For $M < \bar{m}_1$, the $U_q(gl(2|2))$ -weights of both modules $V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ and $\hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ are given by the set $\Xi_{\{\bar{n}_i; \bar{m}_i\}}^{(M)}$.

The level-zero modules related to $X_{\{1\}}^{(M)}$ consist of the finite-dimensional module $W_{y_1, \{M\}}^*$ and the infinite-dimensional module $\hat{V}_{\{M; 1; 1; y_1\}}$. The latter is defined by (170) with the factor $V_{y_1} \otimes V_{y_2} \otimes \dots \otimes V_{y_{\bar{K}}}$ omitted and by (171), (172) without the sum on the rhs of the second line in (172).

In a second step, level-zero modules corresponding to infinite sets of configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ or $(\dots \otimes w_{j_4}^* \otimes w_{j_3} \otimes w_{j_2}^* \otimes w_{j_1})$ are constructed from tensor products $W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ and $W_{x_1, \{n_i; m_i\}} \otimes \hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$. First, for two border stripes $R, M, K, \{n_i; m_i\}$ and $\bar{R}, \bar{M}, \bar{K}, \{\bar{n}_i; \bar{m}_i\}$ with $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$ the product

$$W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}} \quad (173)$$

will be considered. With

$$x_1 = q^{2(\bar{M}-m_R+\delta_{R,1}-1)} y_1 \quad (174)$$

the entire tensor product (173) is annihilated by $\Delta^{(2M+\bar{K}-1)}(H_s^1 + H_s^3)$ for any $s \neq 0$. Its highest weight vector is given by $\nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \equiv \nu_{\{n_i; m_i\}} \otimes \tilde{\nu}_{\{M; \bar{m}_1; y_i\}} = \nu_{\{n_i; m_i\}} \otimes w_3^{* \otimes M} \otimes v_{2,0}^{\otimes \bar{K}}$. The eigenvalues of the coproducts of H_s^j acting on the highest weight vector follow from the highest weight properties of $\nu_{\{n_i; m_i\}}$ and $\tilde{\nu}_{\{M; \bar{m}_1; y_i\}}$:

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(H_s^1) \nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} &= q^{2s(m_R-\delta_{R,1}+1)-sM} \frac{[sM]}{s} x_1^s \nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \\ \Delta^{(2M+\bar{K}-1)}(H_s^2) \nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} &= \frac{1}{s} x_1^s \left\{ q^{-2s(\bar{M}-m_R+\delta_{R,1}-2)} \sum_{i=1}^{\bar{R}-1} q^{2s\bar{t}_i-1-s(\bar{n}_{\bar{R}-i}+1)} [s\bar{n}_{\bar{R}-i}] \right. \\ &\quad \left. - \sum_{i=1}^{R-1} q^{-2s(t_{i-1}-m_R)+s(n_{R-i}+1)} [sn_{R-i}] \right\} \nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \end{aligned} \quad (175)$$

Here (174) is taken into account. For $\bar{M} = 1$, $m_R = \delta_{R,1}$ and $R, \bar{R} \leq 2$, a highest weight vector with $U_q(gl(2|2))$ -weight $(M - n_{R-1} - 1, -K + n_{R-1}, -M + n_{R-1} + 1, 2M - K - n_{R-1} - 2)$ is present in (173) with x_1 and y_1 related by (174). The quotient of $W_{x_1, \{n_i; m_i\}} \otimes W_{x_1, \{M\}}^*$ by the $U_q(\widehat{sl}(2|2))$ -module generated on this highest weight vector may be called $W_{x_1, \{n_i; m_i\}, M}$. If $m_R \neq \delta_{R,1}$, a vector with $U_q(gl(2|2))$ -weight $(M - 1, -K, -M + 1, 2M - K - 2)$ is found in (173), (174) but not among the vectors obtained by applying the $U_q(\widehat{sl}(2|2))$ -generators on the highest weight vector $\nu_{\{n_i; m_i\}} \otimes w_3^{* \otimes M}$. In this case, $W_{x_1, \{n_i; m_i\}, M}$ denotes the module generated on $\nu_{\{n_i; m_i\}} \otimes w_3^{* \otimes M}$ by the level-zero action of $U_q(\widehat{sl}(2|2))$.

Generally, the $U_q(\widehat{sl}(2|2))$ -module created on the highest weight vector $\nu_{\{n_i; m_i\}} \otimes \tilde{\nu}_{\{M; \bar{m}_1; y_i\}} \in W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with (174) may contain only a part of the $U_q(\widehat{sl}(2|2))$ -weights of the corresponding part in $\{\tau\}_{n_{\{n_i; m_i\}}} \otimes \{\tau^*\}_{\bar{n}_{\{\bar{n}_i; \bar{m}_i\}}}$. A simple example is provided by the case $R = \bar{R} = M = \bar{M} = 2$. The level-zero action of $U_q(\widehat{sl}(2|2))$ on $\nu_{\{1; 1, 0\}} \otimes \tilde{\nu}_{\{2; 1; q^{-2} x_1\}} \in W_{x_1, \{1; 1, 0\}} \otimes V_{\{2; 2; 1; q^{-2} x_1\}}$ does not yield a vector $a f_2 \nu_{\{1; 1, 0\}} \otimes \tilde{\nu}_{\{2; 1; q^{-2} x_1\}} + b \nu_{\{1; 1, 0\}} \otimes f_2 \tilde{\nu}_{\{2; 1; q^{-2} x_1\}}$ with $a \neq q^{-1} b$. The $U_q(gl(2|2))$ -weight structure of the $U_q(\widehat{sl}(2|2))$ -module $V_{x_1, \{1; 1, 0\}, \{1; 1, 0\}}$ generated from such a vector coincides with the weight structure of the part of $\{\tau\}_{-2} \otimes \{\tau^*\}_{-1}$ associated to two border stripes $\{1; 1, 0\}$. For general $R, \bar{R} > 1$, there are $\binom{L}{L'}$ vectors with $U_q(gl(2|2))$ -weight $(M - L', \bar{K} - K, -M + L', 2M - K - \bar{K})$ in $W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ missing among the vectors generated from all vectors with weight $(M - L' + 1, \bar{K} - K, -M + L' - 1, 2M - K - \bar{K})$ if

$$q^{-2(t_{i_l} - m_R)} x_1 = q^{2\bar{t}_{j_l}} y_1 \quad (176)$$

with $1 \leq l \leq L < \min(R, \bar{R})$, $1 \leq L' \leq L$, $0 \leq i_l \leq R - 2$, $0 \leq j_l \leq \bar{R} - 2$ and $i_l \neq i_{l'}, j_l \neq j_{l'}$ for $l \neq l'$. In (176), t_i is defined by (118), $\bar{t}_0 = 0$ and $\bar{t}_j = \sum_{j'=1}^j (\bar{n}_{\bar{R}-j'} + \bar{m}_{\bar{R}-j'})$, $1 \leq j \leq \bar{R} - 2$. These may be regarded as vectors in the quotient of $W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ by the $U_q(\widehat{sl}(2|2))$ -modules generated on the vectors with $U_q(gl(2|2))$ -weight $(M - L' + 1, \bar{K} - K, -M + L' - 1, 2M - K - \bar{K})$ for a given L' . This allows to write them as linear combinations of eigenvectors $\nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})}$ of $\Delta^{(2M+K-1)}(H_s^j)$, $j = 1, 2, 3$:

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(H_s^1) v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})} &= \frac{1}{s} x_1^s q^{2s(m_R+1)} \left(q^{-sM} [Ms] - \sum_{l=1}^{L'} q^{-s(2t_{r_l}+1)} [s] \right) v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})} \\ &= -\Delta^{(2M+\bar{K}-1)}(H_s^3) v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})} \\ \Delta^{(2M+\bar{K}-1)}(H_s^2) v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})} &= \frac{1}{s} x_1^s \left\{ q^{-2s(\bar{M}-m_R+\delta_{R,1}-2)} \sum_{i=1}^{\bar{R}-1} q^{2s\bar{t}_{i-1}-s(\bar{n}_{\bar{R}-i}+1)} [s\bar{n}_{\bar{R}-i}] \right. \\ &\quad \left. - \sum_{i=1}^{R-1} q^{-2s(t_{i-1}-m_R)+s(n_{R-i}+1)} [sn_{R-i}] \right\} v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L'; r_1, r_2, \dots, r_{L'})} \end{aligned} \quad (177)$$

where the parameters r_l are L' different numbers chosen in $\{i_{l'}\}_{1 \leq l' \leq L}$. The $U_q(\widehat{sl}(2|2))$ -module $V_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L)}$ generated on $v_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}^{(L; l_1, l_2, \dots, l_L)}$ will be denoted by $\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$. It contains all vectors in $W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with $U_q(gl(2|2))$ -weights $(M - \hat{L}, \bar{K} - K, -M + \hat{L}, 2M - K - \bar{K})$, $\hat{L} = 0, 1, 2, \dots$

For a pair of border stripes not satisfying any condition of the form (176), $\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ denotes the $U_q(\widehat{sl}(2|2))$ -module obtained on $\nu_{\{n_i; m_i\}} \otimes w_3^{* \otimes M} \otimes v_{2,0}^{\otimes \bar{K}} \in W_{x_1, \{n_i; m_i\}} \otimes V_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with relation (174). In particular, the equations (176) are incompatible with the requirement (174) if $M < \bar{m}_1$. Explicit examination reveals that in the cases $1 \leq R, \bar{R} \leq 2$ with $\bar{M} > 1$ and $K = 0, 1$ for $m_R \neq \delta_{R,1}$ the $U_q(gl(2|2))$ -weights of $\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ exactly correspond to the weights of the set $\Xi_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ if $\bar{M} \geq \bar{m}_1$.

If $\bar{M} = 1$ or $M < \bar{m}_1$, the tensor product

$$W_{x_1, \{n_i; m_i\}} \otimes \hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \quad (178)$$

remains to be considered. Provided that x_1 and y_1 are related by (174), some general statements on the structure of the product (178) can be formulated. It contains a vector $\nu_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ with $U_q(gl(2|2))$ -weight $(0, \bar{K} - K +$

$M, 0, M - K - \bar{K} - 2$) which does not result from the level-zero action of $U_q(\widehat{sl}(2|2))$ on $\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$. The vector is characterized by

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(H_s^1) \dot{\nu}_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} &= \Delta^{(2M+\bar{K}-1)}(H_s^3) \dot{\nu}_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} = 0 \\ \Delta^{(2M+\bar{K}-1)}(H_s^2) \dot{\nu}_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} &= \\ &= \frac{[s]}{s} \left\{ q^{s(2\bar{M}-1)} y_1^s \left(q^{-sm_R} [sm_R] + \sum_{i=1}^{R-2} q^{-s(2t_i+m_{R-i-1})} [sm_{R-i-1}] \right) + q^{2s} \sum_{i=1}^{\bar{K}} y_i^s \right\} \dot{\nu}_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \end{aligned} \quad (179)$$

The level-zero action of $U_q(\widehat{sl}(2|2))$ on $\dot{\nu}_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ provides the whole tensor product (178). Due to (174), both $\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ and $\Delta^{(2M+\bar{K}-1)}(f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}})$ satisfy the highest weight properties. For the latter they read

$$\Delta^{(2M+\bar{K}-1)}(E_s^{l,+} \cdot f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) = 0 \quad \forall s, l = 1, 2, 3 \quad (180)$$

$$\Delta^{(2M+\bar{K}-1)}(H_s^1 \cdot f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) = \Delta^{(2M+\bar{K}-1)}(H_s^3 \cdot f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) = 0$$

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(H_s^2 \cdot f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) &= \\ &= \frac{[s]}{s} \left\{ q^{s(2\bar{M}-1)} y_1^s \left(q^{-sm_R} [sm_R] + \sum_{i=1}^{R-2} q^{-s(2t_i+m_{R-i-1}-\delta_{i,R-2})} [s(m_{R-i-1} - \delta_{i,R-2})] \right) + \right. \\ &\quad \left. + q^{2s\bar{M}} y_1^s + q^{2s} \sum_{i=1}^{\bar{K}} y_i^s \right\} \Delta^{(2M+\bar{K}-1)}(f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) \end{aligned} \quad (181)$$

with $s \neq 0$. The vector $\Delta^{(M-1)}(f_1) \nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ is annihilated by $\Delta^{(2M+\bar{K}-1)}(H_s^1)$ and $\Delta^{(2M+\bar{K}-1)}(H_s^3)$. Moreover, it satisfies

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(E_s^{l,+}) \left(\Delta^{(M-1)}(f_1) \nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \right) &= \\ &= \delta_{l,1} q^{2s\bar{M}} y_1^s [M] \cdot \Delta^{(2M+\bar{K}-1)}(f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}) \end{aligned} \quad (182)$$

for $l = 1, 2, 3$ and

$$\begin{aligned} \Delta^{(2M+\bar{K}-1)}(H_s^2) \left(\Delta^{(M-1)}(f_1) \nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \right) &= \\ &= \frac{[s]}{s} \left\{ q^{s(2\bar{M}-1)} y_1^s \left(q^{-sm_R} [sm_R] + \sum_{i=1}^{R-2} q^{-s(2t_i+m_{R-i-1}-\delta_{i,R-2})} [s(m_{R-i-1} - \delta_{i,R-2})] \right) + q^{2s} \sum_{i=1}^{\bar{K}} y_i^s \right\} \cdot \\ &\quad \cdot \left(\Delta^{(M-1)}(f_1) \nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}} \right) \end{aligned} \quad (183)$$

Thus $\Delta^{(M-1)}(f_1) \nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1) \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ is a highest weight vector in the quotient of $W_{x_1, \{n_i; m_i\}} \otimes \hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ by the $U_q(\widehat{sl}(2|2))$ -module generated on $\Delta^{(2M+\bar{K}-1)}(f_1) (\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}})$. At $\bar{M} = 1$, the last sum on the rhs of equations (179), (181) and (183) is dropped.

These observations allow to specify particular level-zero modules contained in (178) with (174). The cases $M < \bar{m}_1$ and $\bar{M} = 1$ need to be distinguished.

If $\bar{M} = 1$, $\hat{V}_{x_1, \{n_i; m_i\}, M}$ is defined as the level-zero $U_q(\widehat{sl}(2|2))$ -module generated on $\nu_{\{n_i; m_i\}} \otimes \hat{\nu}_{\{M; 1; 1; y_i\}}$. For $1 \leq R \leq 2$, the $U_q(gl(2|2))$ -weights found in $W_{x_1, \{n_i; m_i\}, M}$ and $\hat{V}_{x_1, \{n_i; m_i\}, M}$ are given by the weights of the set $\Xi_{\{n_i; m_i\}, \{1\}}$.

In the case $M < \bar{m}_1$, the quotient of $W_{x_1, \{n_i; m_i\}} \otimes \hat{V}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ with (174) by the $U_q(\widehat{sl}(2|2))$ -module generated on $\Delta^{(M-1)}(f_1)\nu_{\{n_i; m_i\}} \otimes \Delta^{(M+\bar{K}-1)}(f_1)\hat{\nu}_{\{M; \bar{M}; \bar{m}_1; y_i\}}$ will be denoted by $\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$. For $M < \bar{m}_1$ and $1 \leq R, \bar{R} \leq 2$ it is readily verified that $\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ and $\tilde{\hat{V}}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ together contain exactly the same $U_q(gl(2|2))$ -weights as the set $\Xi_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$. Formally, the statement can be extended to the parameters $R = 0$, $\{\bar{n}_i; \bar{m}_i\}$. Then the related modules (163) and (164) are denoted by $\tilde{V}_{x_1, \{\bar{n}_i; \bar{m}_i\}}$ and $\tilde{\hat{V}}_{x_1, \{\bar{n}_i; \bar{m}_i\}}$, respectively. Here the spectral parameter x_1 is fixed by $x_1 = q^{2(\bar{M}-1)} y_1$. $\tilde{V}_{x_1, \{1\}}$ is the one-dimensional module with $U_q(gl(2|2))$ -weight $(0, 0, 0, 0)$.

It seems that the modules constructed from the tensor products (173) and (178) have not been considered before. In general, they are reducible as $U_q(sl(2|2))/(h_1 + h_3)$ -modules. The $sl(2|2)$ -modules associated to the irreducible components are found in [40] and some of the references therein.

The above statements on the weight structures of the level-zero modules may be assumed to hold true for all R and \bar{R} .

Conjecture IV: The $U_q(gl(2|2))$ -weights of the level-zero modules attributed to a pair of border stripes exactly coincide with the weights in the sets $\Xi_{\{\bar{n}_i; \bar{m}_i\}}$ (for $R = 0$) or $\Xi_{\{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$ (for $R > 0$).

Hence the weight-preserving one-to-one correspondence between the labelings $T_{A,B}$ and the space Ω_A of half-infinite configurations implies a decomposition of the $U_q(gl(2|2))$ -weight structure of Ω_A into the weight structures of the level-zero modules.

The pairs are formed by a finite border strip parameterized either by $R = 0$ or by $R > 0$, $\{n_i; m_i\}$ and an infinite border strip with parameters $\bar{R} > 0$, $\{\bar{n}_i; \bar{m}_i\}$, $\bar{m}_{\bar{R}} = \delta_{\bar{R},1}$. A complete list of pairs of border stripes and the related level-zero modules is provided by the following table:

border stripes	level-zero modules of $U_q(\widehat{sl}(2 2))$
$R = 0; \bar{R}, \{\bar{n}_i; \bar{m}_i\}, \bar{m}_{\bar{R}} = \delta_{\bar{R},1}$	$\tilde{V}_{x_1, \{\bar{n}_i; \bar{m}_i\}} \oplus \tilde{\hat{V}}_{x_1, \{\bar{n}_i; \bar{m}_i\}}$
$R, \{n_i; m_i\}; \bar{R}, \{\bar{n}_i; \bar{m}_i\}, m < \bar{m}_1, \bar{M} > 1, \bar{m}_{\bar{R}} = 0$	$\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}} \oplus \tilde{\hat{V}}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$
$R, \{n_i; m_i\}; \bar{R} = \bar{M} = 1$	$W_{x_1, \{n_i; m_i\}, M} \oplus \hat{V}_{x_1, \{n_i; m_i\}, M}$
$R, \{n_i; m_i\}; \bar{R}, \{\bar{n}_i; \bar{m}_i\}, M \geq \bar{m}_1, \bar{M} > 1, \bar{m}_{\bar{R}} = 0$	$\tilde{V}_{x_1, \{n_i; m_i\}, \{\bar{n}_i; \bar{m}_i\}}$

All modules except $\tilde{V}_{x_1, \{1\}}$ and $W_{x_1, \{n_i; m_i\}, M}$ are infinite-dimensional. The labelings $T_{A,B}$ consist of a restricted type A labeling of the finite border strip and a type B labeling of the infinite border strip. Ω_A is the space of all half-infinite configurations $(\dots \otimes w_{j_4} \otimes w_{j_3}^* \otimes w_{j_2} \otimes w_{j_1}^*)$ with $j_r = 3$ for almost all r . Here the $U_q(gl(2|2))$ -weights of the configurations are determined with respect to the reference weight \bar{h}^A specified in (55). The statement also applies to the space Ω_B of half-infinite configurations with the reference weight \bar{h}^B given in (58).

The modules in this table have been introduced with respect to the evaluation action. Another level-zero action of $U_q(\widehat{sl}(2|2))$ is furnished by a suitable generalization of the q -deformation of the Yangian action given in [22,28] for $U_q(\widehat{sl}(N))$ -models. With respect to this action, the module $\tilde{V}(\Lambda_0)$ may decompose into the level-zero modules collected in the above table. A similar decomposition has been constructed in [26] in the $U_q(\widehat{sl}(N))$ -case. Details for the $U_q(\widehat{sl}(2|2))$ -model studied here will be published separately.

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- [1] R.J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London, 1982
- [2] Date E., Jimbo M., Kuniba A., Miwa T. and Okado M., One dimensional configuration sums in vertex models and affine Lie algebra characters, *Lett. Math. Phys.* 17 (1989) 69-77
- [3] Foda O. and Miwa T., Corner transfer matrices and quantum affine algebras, *Int. J. Mod. Phys. A* 7 Supplement 1A (1992) 279-302
- [4] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, Diagonalization of the XXZ Hamiltonian by vertex operators, *Comm. Math. Phys.* 151 (1993) 89-153
- [5] M. Jimbo, T. Miwa, Algebraic Analysis of Solvable Lattice Models, Regional Conference Series in Mathematics, Vol. 85, AMS, 1995
- [6] J. Hong, S.-J. Kang, T. Miwa and R. Weston, Mixing of Ground States in Vertex Models, *J. Phys. A* 31 (1998) L515-L525
- [7] J. Hong, S.-J. Kang, T. Miwa and R. Weston, Vertex Models with Alternating Spins, *Asian J. Math.* 2 (1998) 711-758
- [8] Andrews G.E., Baxter R.J. and Forrester P.J., Eight vertex SOS model and generalized Rogers-Ramanujan identities, *J. Stat. Phys.* 35 (1984) 193-266
- [9] Date E., Jimbo M., Kuniba A., Miwa T. and Okado M., Exactly solvable SOS models: I. Local height probabilities and theta function identities, *Nucl. Phys. B* 290[FS20] (1987) 231-273
- [10] Date E., Jimbo M., Kuniba A., Miwa T. and Okado M., Exactly solvable SOS models: II. Proof of the star-triangle relation and combinatorial identities, *Adv. Stud. in Pure Math.* 16 (1988) 17-122
- [11] M. Jimbo, T. Miwa and Y. Ohta, Structure of the space of states in $RSOS$ models, *Int. J. Mod. Phys. A* 8 (1993) 1457-1477
- [12] R.M. Gade, The $U_q(\widehat{sl}(2|1))_1$ -module $V(\Lambda_2)$ and a corner transfer matrix at $q = 0$, *Nucl. Phys. B* 659 [PM] (2003) 387-423
- [13] I. Cherednik, A new interpretation of Gelfand-Zetlin bases, *Duke Math. J.* 54 (1987) 563-577
- [14] M. Narasov and V. Tarasov, Representations of Yangians with Gelfand-Zetlin bases, *J. Reine Angew. Math.* 496 (1998) 181-212
- [15] M. Narasov and V. Tarasov, On irreducibility of Tensor Products of Yangian modules, *Internat. Math. Research Notices* (1998) 125-150
- [16] A. Kirillov, A. Kuniba and T. Nakanishi, Skew Young Diagram method in spectral decomposition of integrable lattice models, *Comm. Math. Phys.* 185 (1997) 441-465
- [17] T. Arakawa, T. Nakanishi, K. Oshima and A. Tsuchiya, Spectral decomposition of path space in solvable lattice models, *Comm. Math. Phys.* 181 (1996) 159-182
- [18] A. Kirillov, A. Kuniba and T. Nakanishi, Skew Young diagram method in spectral decomposition of integrable lattice models II: Higher levels, *Nucl. Phys. B* 529 (1998) 611-638
- [19] D. Uglov, Yangian actions on higher level irreducible integrable modules of $\widehat{\mathfrak{gl}}_N$, preprint math.QA/9802048 (1998)
- [20] F.D.M. Haldane, Z.N.C. Ha, J.C. Testra, D. Bernard and V. Pasquier, Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory, *Phys. Rev. Lett.* 69 (1992) 2021-2024
- [21] D. Bernard, V. Pasquier and D. Serban, Spinons in conformal field theory, *Nucl. Phys. B* 428 (1994) 612-628
- [22] M. Jimbo, R. Kedem, H. Konno, T. Miwa and J.-U. H. Petersen, New level-0 action of $U_q(\widehat{sl}_2)$ on level-1 modules, *Proc. Statistical Mechanics and Quantum Field theory, USC*, 1994
- [23] M. Jimbo, R. Kedem, H. Konno, T. Miwa and J.-U. H. Petersen, Level-zero action of $U_q(\widehat{sl}_2)$ on level-1 modules and Macdonald Polynomials, *J. Phys. A* 28 (1995) 5589
- [24] Y. Saito, Quantum toroidal algebras and their vertex representations, *Publ. RIMS* 34 (1998) 155-177
- [25] Y. Saito, K. Takemura and D. Uglov, Toroidal actions on level-1 modules of $U_q(\widehat{sl}_n)$, *Transformation Groups* 3 (1998) 75-102
- [26] K. Takemura, The decomposition of level 1 irreducible highest weight modules with respect to the level 0 actions of the quantum affine algebra $U_q(\widehat{sl}_n)$, *J. Phys. A* 31 (1998) 1467-1485
- [27] M. Varagnolo and E. Vasserot, Double-loop algebras and the Fock space, *Invent. Math.* 133 (1998) 133-159
- [28] K. Takemura and D. Uglov, Level 0 action of $U_q(\widehat{sl}_n)$ on the q -deformed Fock spaces, *Comm. Math. Phys.* 190 (1998) 549-583
- [29] K. Takemura and D. Uglov, Representations of the Quantum Toroidal Algebra on Highest Weight Modules of the Quantum Affine Algebra of Type \mathfrak{gl}_N , *Publ. RIMS* 35 (1999) 407-450
- [30] V. G. Kac, Infinite-dimensional algebras, Dedekind's η -function, classical Möbius function and the very strange formula, *Adv. Math.* 30 (1978) 85-136
- [31] H. Yamane, On defining relations of the Lie superalgebras and their quantized universal enveloping superalgebras, *q-alg*/9603015
- [32] S. Eswara Rao and V. Futorny, Irreducible modules for affine Lie superalgebras,
- [33] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Affine crystals and vertex models, *Int. J. Mod. Phys. A* 7, Suppl. 1A (1992) 449-484
- [34] V.G. Kac and M. Wakimoto, Integrable highest weight modules over affine super algebras and number theory, *Progress in Mathematics* 123, Birkhäuser, Boston (1994) 415-456
- [35] V.G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell's function, *Comm. Math. Phys.* 215 (2001) 631-682

- [36] V.G. Kac, Lie superalgebras, *Adv. Math.* 26 (1977) 8-96
- [37] S.-J. Cheng, N. Lam, Infinite-dimensional Lie Superalgebras and Hook Schur functions, *Comm. Math. Phys.* 238 (2003) 95-118
- [38] V.G. Kac, Representations of classical Lie superalgebras, *Lect. Notes Math.* 676, Springer-Verlag (1978) 597-626
- [39] Y. Su and R.B. Zhang, Character and dimensional formulae for finite dimensional irreducible representations of the general linear superalgebra, *math.QA/0403315*
- [40] S.-J. Cheng, N. Lam and R.B. Zhang, Character formula for infinite dimensional unitarizable modules of the general linear superalgebra, *math.RT/0301183*
- [41] V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, *Sov. Math. Dokl.* 36 (1988) 212-6
- [42] R.M. Gade, Universal R-matrix and graded Hopf algebra structure of $U_q(\widehat{gl}(2|2))$, *J. Phys. A* 31 (1998) 4909-4925
- [43] K. Kimura, S. Shiraishi and J. Uchiyama, A level-one representation of the quantum affine superalgebra $U_q(\widehat{sl}(M+1|N+1))$, *Comm. Math. Phys.* 188 (1997) 367-378
- [44] P. Bouwknegt, A. Ceresole, J.G. McCarthy and P. van Nieuwenhuizen, Extended Sugawara construction for the superalgebras $SU(M+1|N+1)$. I. Free field representation and bosonization of super Kac-Moody currents, *Phys. Rev. D* 39 (1989) 2971-2986
- [45] W.-L. Yang and Y.-Z. Zhang, Vertex operators of $U_q(\widehat{gl}(N|N))$ and highest weight representations of $U_q(\widehat{gl}(2|2))$, *J. Math. Phys.* 41 (2000) 2460-2481
- [46] I. Penkov and V. Serganova, Representations of classical Lie superalgebras of type I, *Indag. Math.* 3 (1992) 419-466
- [47] J.H.H. Perk and C.L. Schultz, New families of commuting transfer matrices in q -state vertex models, *Phys. Lett. A* 84 (1981) 407-410
- [48] A. Kato, Y.-H. Quano and J. Shiraishi, Free boson representation of q -vertex operators and their correlation functions, *Comm. Math. Phys.* 157 (1993) 119-137
- [49] Y. Koyama, Staggered polarization of vertex models with $U_q(\widehat{sl}(n))$ -symmetry, *Comm. Math. Phys.* 164 (1994) 277-291
- [50] H. Awata, S. Odake and J. Shiraishi, Free boson realization of $U_q(\widehat{sl}_N)$, *Comm. Math. Phys.* 162 (1994) 61-83
- [51] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2-nd ed., Clarendon Press (1995)
- [52] V. Chari and A. Pressley, Quantum affine algebras, *Comm. Math. Phys.* 142 (1991) 261-283
- [53] I. Dimitrov and I. Penkov, Partially and fully integrable modules over Lie super algebras, *Studies in Advanced Mathematics* (series editor S.-T. Yau) 4 AMS and Internatl. Press (1997) 49-67